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# Optical solitons in fibers: theoretical review

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Theoretical properties of light wave envelope propagation in optical fibers are presented. Generation of bright and dark optical solitons, excitation of modulational instabilities and their applications to optical transmission systems are discussed together with other non-linear effects such as the stimulated Raman process.

## 1.1 Introduction

The envelope of a light wave guided in an optical fiber is deformed by the dispersive (variation of the group velocity as a function of the wavelength) and non-linear (variation of the phase velocity as a function of the wave intensity) properties of the fiber. The dispersive property of the light wave envelope is decided by the group velocity dispersion (GVD) which may be described by the second derivative of the axial wavenumber $k$ $(= 2\pi/\lambda)$ with respect to the angular frequency $\omega$ of the light wave, $\partial^2 k/\partial\omega^2$ $(= k'')$. $k''$ is related to the coefficient the group velocity delay $D$ in ps per deviation of wavelength in nm and per distance of propagation in km, through $k'' = D\lambda^2/(2\pi c)$ where $\lambda$ is the wavelength of the light and $c$ is the speed of light. For a standard fiber, $D$ has a value of approximately $-10$ ps/nm · km for the wavelength of approximately 1.5 $\mu$m. $D$ becomes zero near $\lambda = 1.3\ \mu$m for a standard fiber and near $\lambda = 1.5\ \mu$m for a dispersion-shifted fiber.

The non-linear properties of the light wave envelope are determined by a combination of the Kerr effect (an effect of the increase in refractive index $n$ in proportion to the light intensity) and stimulated Brillouin and Raman scatterings. The Kerr effect is described by the Kerr coefficient $n_2$ $(\simeq 1.2 \times 10^{-22}\ \mathrm{m^2/V^2})$ which gives the change

in the refractive index, $n$, in proportion to the electric field intensity $|E|^2$ of the light wave, $n = n_0 + n_2|E|^2$.

The model equation which describes the light wave envelope in the anomalous dispersion regime, $k'' < 0$, is given by the non-linear Schrödinger (NLS) equation or the cubic Schrödinger equation. Using the complex amplitude $q(x, t)$, the equation is expressed by

$$\mathrm{i}\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^2 q = 0$$

In this equation, $Z$ represents the distance along the direction of propagation, and $T$ represents the time (in the group velocity frame). The second term originates from the GVD, and the third term originates from the Kerr effect.

Hasegawa and Tappert (1973a) were the first to show theoretically that an optical pulse in a dielectric fiber forms a solitary wave based on the fact that the wave envelope satisfies the NLS equation.

However, at that time, neither a dielectric fiber which had small loss, nor a laser which emitted a light wave at the appropriate wavelength ($\simeq 1.5\ \mu$m), was available. Furthermore, the dispersion property of the fiber was not known. Consequently, it was necessary to consider a case where the group dispersion, $k''$, is positive, that is when the coefficient of the second term in the NLS equation is negative, in which case the solitary wave appears as the absence of a light wave (Hasegawa and Tappert, 1973b). Two years prior to the publication of the paper by Hasegawa and Tappert, Zakharov and Shabat (1971) showed that the NLS equation can also be solved using the inverse scattering method in a way analogous to the Korteweg–de Vries equation (Gardner *et al.*, 1967). According to this theory, the properties of the envelope soliton of the NLS equation can be described by the complex eigenvalues of Dirac-type equations, the potential being given by the initial envelope wave form. Because of this, the solitary wave solution derived by Hasegawa and Tappert could, in fact, be called a soliton.

Although the electric field in a fiber has a relatively large magnitude ($\simeq 10^6$ V/m) ( for an optical power of a few hundred mW in a fiber with a cross section of 100 $\mu$m$^2$), the total change in the refractive index $n_2|E|^2$ is still $10^{-10}$, and this seems to be negligibly small. The reason why such a small change in the refractive index becomes important is that the modulation frequency $\Delta\omega$, which is determined by the inverse of the pulsewidth, is much smaller than the frequency of the wave $\omega$ and secondly the GVD, which is produced by $\Delta\omega$, is

also small. For example, the angular frequency $\omega$ for light wave with wavelengths $\lambda$ of 1.5 $\mu$m is $1.2 \times 10^{15}$ s$^{-1}$. If a pulse modulation with a pulsewidth of 10 ps is applied to this light wave, the ratio of the modulation frequency $\Delta\omega$ to the carrier frequency $\omega$ becomes approximately $10^{-4}$. As will be shown later, the amount of wave distortion due to the GVD is proportional to $(\Delta\omega/\omega)^2$ times the coefficient of the group dispersion $k''(= \partial^2 k/\partial\omega^2)$. Consequently, if the group dispersion coefficient is of the order of $10^{-2}$, the relative change in the wavenumber due to the group dispersion becomes comparable with the non-linear change.

Seven years after the prediction by Hasegawa and Tappert, Mollenauer *et al*. (1980) succeeded in the generation and transmission of optical solitons in a fiber for the first time.

The fact that the envelope of a light wave in a fiber can be described by the NLS equation can also be illustrated by the generation of modulational instability when a light wave with constant amplitude propagates through a fiber (Akhmanov *et al*., 1968; Hasegawa and Brinkman, 1980; Anderson and Lisak, 1984; Hermansson and Yevick, 1984). Modulational instability is a result of the increase in the modulation amplitude when a wave with constant amplitude propagates through a non-linear dispersive medium with anomalous dispersion, $k'' < 0$. The origin of modulational instability can be identified by looking at the structure of the NLS equation. If we consider the third term in the NLS equation as a potential which traps the quasi-particle described by the NLS equation, the fact that the potential is proportional to the absolute square of the wavefunction indicates that the potential depth becomes deeper in proportion to the density of the quasi-particle. Consequently, when the local density of the quasi-particle increases, the trapping potential increases further thus enhancing the self-induced increase of the quasi-particle density.

A further interesting fact regarding the nature of an optical soliton in a fiber is the observation of the effect of higher order terms which cannot be described by the NLS equation. The small parameter $(\Delta\omega/\omega)$ used for the derivation of the NLS equation is of the order of $10^{-4}$. Therefore, the description of a process which is of higher order in this small parameter would seem to require an extremely accurate experiment.

An example of this higher order effect is the Raman process which exists within the spectrum of a soliton. When the central spectrum of a soliton acts as a Raman pump, amplifying the lower sideband spectra within the soliton spectra, the frequency spectrum gradually

shifts to the lower frequency side. This effect was first observed in an experiment by Mitschke and Mollenauer (1986), and was theoretically explained by Gordon (1986) in terms of the induced Raman process. Kodama and Hasegawa (1987) have identified that Mollenauer's discovery is due to the higher order term which represents the Raman effect. It is thus shown that the Raman process in the spectrum of a soliton produces a continuous shift of the soliton frequency spectrum to the lower frequency side, without changing the pulse shape.

The self-induced Raman process can be utilised to split two or more solitons which are superimposed in a fiber. The reason why it is possible to detect a process which is dependent on such a small parameter is that the light frequency, which is of order $10^{14}$, is very large and, therefore, even if the perturbation is of the order $10^{-10}$, the modification in the light frequency becomes $10^4$ Hz and is consequently readily observable.

One important application of optical solitons is for a high bit-rate optical transmission system. Since solitons are not distorted by fiber dispersion, they can be transmitted for an extended distance (beyond several thousands of kilometres) only by providing amplification to compensate the fiber loss. Since fibers can be converted to amplifiers by themselves, this property of a soliton can be used to construct an all-optical transmission system. Such a system is much more economical and reliable than a conventional system which requires repeaters involving both photonics and electronics in order to reshape the optical pulse distorted by the fiber dispersion.

In this chapter, a theoretical review of these interesting phenomena is presented. In Section 1.2 the derivation of the model non-linear wave equation for the light wave envelope in a cylindrical dielectric guide is presented starting from the first principle. In Section 1.3 examples of non-linear light wave behavior in a fiber are presented including the optical soliton solution (1.3.1), the modulational instability (1.3.2) and the effect of higher-order terms on the soliton propagation (1.3.3). In Section 1.4, technical issues related to the application of optical solitons to all-optical transmission systems are discussed.

## 1.2 Derivation of the envelope equation for a light wave in a fiber

In this section, by introducing an appropriate scale of coordinates based on the physical setting of a cylindrical dielectric guide

we reduce the Maxwell equation (three-dimensional vector equations) into the NLS equation with the higher-order terms describing the linear and the non-linear dispersion, as well as dissipation effects. The method used here is based on the asymptotic perturbation technique developed by Taniuti (1974), and gives a consistent scheme for the derivation of the NLS equation and the higher-order terms. The derivation follows closely Kodama and Hasegawa (1987).

The electric field $\boldsymbol{E}$, with the dielectric constant $\varepsilon_0\chi$, satisfies the Maxwell equation,

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{D} \tag{1.1}$$

Here, the displacement vector $\boldsymbol{D} = \chi_* \boldsymbol{E}$ is the Fourier transform of $\hat{\boldsymbol{D}}(\omega)$ defined by

$$\chi_* \boldsymbol{E}(t) = \int_{-\infty}^{t} \mathrm{d}t_1\, \chi^{(0)}(t - t_1)\boldsymbol{E}(t_1)$$

$$+ \int_{-\infty}^{t}\mathrm{d}t_1 \int_{-\infty}^{t}\mathrm{d}t_2 \int_{-\infty}^{t} \mathrm{d}t_3 \cdot \chi^{(2)}(t - t_1, t - t_2, t - t_3) \cdot \{\boldsymbol{E}(t_1)\cdot\boldsymbol{E}(t_2)\}\boldsymbol{E}(t_3)$$

$$+ \text{(higher non-linear terms)} \tag{1.2}$$

Here, $*$ indicates convolution integral, the second term describing the non-linear polarisation includes the Kerr and Raman effects with proper retardation. The dielectric tensors $\chi^{(0)}$, $\chi^{(2)}$ are dependent on the spatial coordinates in the transverse direction of the fiber axis. We write equation (1.1) in the following form:

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{D} = \nabla(\nabla \cdot \boldsymbol{E}) \tag{1.3}$$

It should be noted that $\nabla \cdot \boldsymbol{E}$ in equation (1.3) is not zero, since $\nabla \cdot \boldsymbol{D} = 0$ (the constraint for $\boldsymbol{D}$ in Maxwell's equation) implies that $\chi_*(\nabla \cdot \boldsymbol{E}) = -(\nabla\chi_*)\cdot \boldsymbol{E} \neq 0$, namely, the electric field cannot be described by either the TE or TM modes.

Since our purpose is to reduce (1.3) in the sense of an asymptotic perturbation it is convenient to write it in the following matrix form,

$$L\mathbf{E} = 0 \tag{1.4}$$

where $\mathbf{E}$ represents a column vector, i.e. $(E_r, E_\theta, E_z)^t$. In cylindrical coordinates, where the $z$-axis is the axial direction of the fiber, the matrix L consisting of the three parts $L = L_\mathrm{a} + L_\mathrm{b} - L_\mathrm{c}$ is defined by

$$L_a = \begin{pmatrix} \nabla_\perp^2 - \frac{1}{r^2} & -\frac{2}{r^2}\frac{\partial}{\partial\theta} & 0 \\ \frac{2}{r^2}\frac{\partial}{\partial\theta} & \nabla_\perp^2 - \frac{1}{r^2} & 0 \\ 0 & 0 & \nabla_\perp^2 \end{pmatrix} \tag{1.5}$$

$$L_b = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\cdot *\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.6}$$

$$L_c = \begin{vmatrix} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r & \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta} & \frac{\partial^2}{\partial r\partial z} \\ \frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}r & \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial z}r & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} & \frac{\partial^2}{\partial z^2} \end{vmatrix} \tag{1.7}$$

It should be noted that these matrices imply that

$$L_a\mathbf{E} = \nabla_\perp^2\mathbf{E} \equiv \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)E$$

$$L_b\mathbf{E} = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E \qquad \text{and} \qquad L_c\mathbf{E} = \nabla(\nabla\cdot E)$$

We consider the electric field as an almost monochromatic wave propagating along $z$-axis with the wavenumber $k_0$ and angular frequency $\omega_0$, i.e. the field $\mathbf{E}$ is assumed to be in the expansion form,

$$\mathbf{E}(r, \theta, z, t) = \sum_{l=-\infty}^{\infty} \mathbf{E}_l(r, \theta, \xi, \tau; \varepsilon)\exp\{\mathrm{i}(k_l z - \omega_l t)\} \tag{1.8}$$

with $\mathbf{E}_{-l} = \mathbf{E}_l^*$ (complex conjugate). $k_l = lk_0$, $\omega_l = l\omega_0$ and the summation is taken over all the harmonics generated by the non-linear response of the polarisation, $E_l(r, \theta, \xi, \tau; \varepsilon)$ being the envelope of the $l$th harmonic which changes slowly in $z$ and $t$. Here, the slow variables $\xi$ and $\tau$ are defined by

$$\xi = \varepsilon^2 z \qquad \tau = \varepsilon\left(t - \frac{z}{v_g}\right) \tag{1.9}$$

where $v_g$ is the group velocity of the wave given later. Since the radius of the fiber has the same order as the wavelength $(2\pi/k_0)$, the

scale for the transverse coordinates $(r, \theta)$ is of order 1. In this scale of the coordinates of equation (1.9), the behavior of the field can be deduced from the balance between the non-linearity and dispersion which results in the formation of optical solitons confined in the transverse direction.

If we proceed from equations (1.2) and (1.9) we find that the displacement, $\boldsymbol{D} = \chi_* \boldsymbol{E} = \sum \boldsymbol{D}_l \exp\{\mathrm{i}k_l z - \omega_l t)\}$, is given by

$$\begin{aligned}\mathbf{D}_l = {} & \chi_l^{(0)}\mathbf{E}_l + \varepsilon \mathrm{i} \frac{\partial \chi_l^{(0)}}{\partial \omega_l} \frac{\partial \mathbf{E}_l}{\partial \tau} - \varepsilon^2 \frac{1}{2} \frac{\partial^2 \chi_l^{(0)}}{\partial \omega_l^2} \frac{\partial^2 \mathbf{E}_l}{\partial \tau^2} \\ & - \varepsilon^3 \frac{\mathrm{i}}{6} \frac{\partial^3 \chi_l^{(0)}}{\partial \omega_l^3} \frac{\partial^3 \mathbf{E}_l}{\partial \tau^3} + \sum_{l_1+l_2+l_3=l} \{\chi^{(2)}_{l_1 l_2 l_3} (\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2}) \mathbf{E}_{l_3} \\ & + \mathrm{i} \sum_{i=1}^{3} \left( \frac{\partial}{\partial \omega_{l_i}} \chi^{(2)}_{l_1 l_2 l_3} \right) \frac{\partial}{\partial \tau_i} (\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2}) \mathbf{E}_{l_3}\} + \ldots \end{aligned} \tag{1.10}$$

where $\chi_l^{(0)}$ is the Fourier coefficient $\hat{\chi}^{(0)}(\Omega)$ of $\chi^{(0)}(t)$ at $\Omega = \omega_l$, i.e. $\chi_l^{(0)} = \hat{\chi}^{(0)}(\omega_l)$, and $\chi^{(2)}_{l_1 l_2 l_3}$ is the Fourier coefficient $\hat{\chi}^{(2)}(\Omega_1, \Omega_2, \Omega_3)$ of $\chi^{(2)}(t_1, t_2, t_3)$ at $\Omega_1 = \omega_{l_1}$, $\Omega_2 = \omega_{l_2}$, $\Omega_3 = \omega_{l_3}$, and

$$\partial(\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2})\mathbf{E}_{l_3}/\partial \tau_1 = (\partial \mathbf{E}_{l_1}/\partial \tau \cdot \mathbf{E}_{l_2})\mathbf{E}_{l_3}$$

$$\partial(\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2})\mathbf{E}_{l_3}/\partial \tau_2 = (\mathbf{E}_{l_1} \cdot \partial \mathbf{E}_{l_2}/\partial \tau)\mathbf{E}_{l_3}$$

and so on. The last term in equation (1.10) represents the retarded response of the non-linear polarisation which gives both the higher-order non-linear dispersion and dissipation. We note that owing to the dispersion properties of the dielectric constant, the real space wave equation contains terms with higher-order time derivatives. From equations (1.9) and (1.10), equation (1.4) can be written in the following expansion form:

$$\begin{aligned} & L_l \mathbf{E}_l + \mathrm{i}\varepsilon \frac{\partial L_l}{\partial \omega_l} \frac{\partial \mathbf{E}_l}{\partial \tau} - \frac{\varepsilon^2}{2} \frac{\partial^2 L_l}{\partial \omega_l^2} \frac{\partial^2 \mathbf{E}_l}{\partial \tau^2} - \frac{\mathrm{i}}{6} \varepsilon^3 \frac{\partial^3 L_l}{\partial \omega_l^3} \frac{\partial^3 \mathbf{E}_l}{\partial \tau^3} \\ & - \varepsilon^2 \left\{ \frac{\partial}{\partial k_l} \left( \mathbf{L}_l - \frac{\omega_l^2 \chi_l^{(0)}}{c^2} \right) \right\} \cdot \left( \mathrm{i} \frac{\partial \mathbf{E}_l}{\partial \xi} - \frac{1}{2} \frac{\partial^2 k_l}{\partial \omega_l^2} \frac{\partial^2 \mathbf{E}_l}{\partial \tau^2} - \varepsilon \frac{\mathrm{i}}{6} \frac{\partial^3 k_l}{\partial \omega_l^3} \frac{\partial^3 \mathbf{E}_l}{\partial \tau^3} \right) \\ & - \varepsilon^3 \left\{ \frac{\partial^2}{\partial \omega_l \partial k_l} \left( L_l - \frac{\omega_l^2 \chi_l^{(0)}}{c^2} \right) \right\} \mathrm{i} \frac{\partial}{\partial \tau} \left( \mathrm{i} \frac{\partial \mathbf{E}_l}{\partial \xi} - \frac{1}{2} \frac{\partial^2 k_l}{\partial \omega_l^2} \frac{\partial^2 \mathbf{E}_l}{\partial \tau^2} \right) \end{aligned}$$

*continued overleaf*

$$+ \frac{1}{c^2} \sum_{l_1+l_2+l_3=l} \Bigg[ \omega_l^2 \chi_{l_1 l_2 l_3}^{(2)} (\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2}) \mathbf{E}_{l_3}$$

$$+ \mathrm{i}\varepsilon \sum_{i=1}^{3} \left\{ \frac{\partial}{\partial \omega_{l_i}} (\omega_l^2 \chi_{l_1 l_2 l_3}^{(2)}) \right\} \frac{\partial}{\partial \tau_i} (\mathbf{E}_{l_1} \cdot \mathbf{E}_{l_2}) \mathbf{E}_{l_3} \Bigg]$$

$$+ \mathrm{i}\varepsilon \left( \frac{1}{v_g} - \frac{\partial k_l}{\partial \omega_l} \right) \left\{ \frac{\partial}{\partial k_l} \left( L_l - \frac{\omega_l^2 \chi_l^{(0)}}{c^2} \right) \right\} \frac{\partial \mathbf{E}_l}{\partial \tau}$$

$$+ \varepsilon^2 \left( \frac{1}{v_g} - \frac{\partial k_l}{\partial \omega_l} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial \tau} \left\{ \left( \frac{1}{v_g} + \frac{\partial k_l}{\partial \omega_l} \right) \frac{\partial \mathbf{E}_l}{\partial \tau} - 2\varepsilon \frac{\partial \mathbf{E}_l}{\partial \xi} \right\}$$

$$+ \ldots = 0 \tag{1.11}$$

where $L_l$ is $L$ in equations (1.5) to (1.7) with the replacement $\partial/\partial z = \mathrm{i}k_l$, $\partial/\partial t = -\mathrm{i}\omega_l$ and $\chi_* = \chi_l^{(0)}$. It should be noted that the operator $L_l$ is self-adjoint in the sense of the following inner product:

$$(\mathbf{U}, L\mathbf{V}) \equiv \int \mathbf{U}^* \cdot L\mathbf{V} \mathrm{d}S$$

$$= \int L^* \mathbf{U}^* \cdot \mathbf{V} \mathrm{d}S = (L\mathbf{U}, \mathbf{V}) \tag{1.12}$$

where $\mathrm{d}S = r\mathrm{d}r\mathrm{d}\theta$ and $\mathbf{U}, \mathbf{V} \to 0$ as $\|x_\perp\| \to \infty$.

We now assume that $\mathbf{E}_l(r, \theta, \xi, \tau; \varepsilon)$ can be expanded in terms of $\varepsilon$,

$$\mathbf{E}_l(r, \theta, \xi, \tau; \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{E}_l^{(n)}(r, \theta, \xi, \tau) \tag{1.13}$$

Then, from equations (1.4), (1.8), (1.9) and (1.10) we have, at order $\varepsilon$,

$$L_l \mathbf{E}_l^{(1)} = 0 \tag{1.14}$$

In equation (1.14) we consider a mono-mode fiber in which there is only one bound state with eigenvalue $k_0^2$ (i.e. $l = \pm 1$) and the eigenfunction $\mathbf{U} = \mathbf{U}(r, \theta)$ (which is the mode-function describing the confinement of the pulse in the transverse direction and, in general, consists of two parts corresponding to the right and left polarisations). We further assume that the fiber maintains the polarisation (a polarisation preserving fiber). Without this assumption the resultant equation for the wave envelope becomes a coupled equation between the left and the right polarised waves. The solution to equation (1.14) may then be written as

$$\mathbf{E}_l^{(1)}(r, \theta, \xi, \tau) = \begin{cases} q_1^{(1)}(\xi, \tau)\mathbf{U}(r, \theta) & \text{for } l = 1 \\ 0, & \text{for } l \neq \pm 1 \end{cases} \tag{1.15}$$

Here, the coefficient $q_1^{(1)}(\xi, \tau)$ with $q_{-1}^{(1)} = q_1^{(1)*}$ is a complex, scalar function satisfying certain equations determined by the higher-order equation of equation (1.11). From the expression $L_1\mathbf{U} = 0$, the inner product $(\mathbf{U}, L_1\mathbf{U}) = 0$ gives the linear dispersion relation $k_0 = k_0(\omega_0)$,

$$\frac{1}{4}k_0^2 S_0 = \frac{\omega_0^2}{c^2}(\mathbf{U}, n_0^2\mathbf{U}) + (\mathbf{U}, L_0\mathbf{U}) + \mathrm{i}k_0\int(U_z\nabla_\perp\cdot\mathbf{U}^* - U_z^*\nabla_\perp\cdot\mathbf{U})\mathrm{d}S \tag{1.16}$$

where $L_0 = L_l(l = 0)$, $n_0 = (\chi_0^{(0)})^{1/2}$ is the refractive index, and we have assumed the normalisation for $\mathbf{U}$ to be $\int(|U_r|^2 + |U_\theta|^2)\mathrm{d}S = S_0/4$.

At order $\varepsilon^2$, we have

$$L_1\mathrm{E}_1^{(2)} = \mathrm{i}\left\{-\frac{\partial L_l}{\partial\omega_l} - \left(\frac{1}{v_g} - \frac{\partial k_0}{\partial\omega_0}\right)\frac{\partial}{\partial k_l}\cdot\left(L_l - \frac{\omega_l^2}{c^2}\chi_l^{(0)}\right)\right\}\frac{\partial\mathrm{E}_l^{(1)}}{\partial\tau} \tag{1.17}$$

from which we obtain $\mathbf{E}_l^{(2)} = 0$ if $l \neq \pm 1$. In the case where $l = 1$, it is necessary that the inhomogeneous equation (1.17) satisfies the compatibility condition (the condition required for equation (1.17) to be solvable, which is known as the Fredholm alternative)

$$(\mathbf{U}, L_1\mathbf{E}_1^{(2)}) = 0 \tag{1.18}$$

This gives the group velocity $v_g$ in terms of the linear dispersion relation equation (1.16)

$$\frac{1}{v_g} = \frac{\partial k_0}{\partial\omega_0} \tag{1.19}$$

and for $l = 1$ equation (1.17) becomes

$$L_1\mathbf{E}_1^{(2)} = -\mathrm{i}\frac{\partial L_1}{\partial\omega_0}\frac{\partial\mathbf{E}_1^{(1)}}{\partial\tau} = -\mathrm{i}\frac{\partial L_1}{\partial\omega_0}\frac{\partial q_1^{(1)}}{\partial\tau}\mathbf{U} \tag{1.20}$$

From equation (1.13) for $l = 1$, the solution of equation (1.20) may be found in the form

$$\mathbf{E}_1^{(2)} = \mathrm{i}\frac{\partial q_1^{(1)}}{\partial\tau}\frac{\partial\mathbf{U}}{\partial\omega_0} + q_1^{(2)}\mathbf{U} \tag{1.21}$$

where $q_1^{(2)} = q_1^{(2)}(\xi, \tau)$ with $q_{-1}^{(2)} = q_1^{(2)*}$ is a scalar function to be determined in the higher-order equation. As we will see later, the first term in equation (1.21) represents the effect of wave guide dispersion in the coefficient of the non-linear dispersion terms in the NLS equation.

At order $\varepsilon^3$, we have

$$L_l\mathbf{E}_l^{(3)} \begin{cases} = 0 \qquad \text{if } l \neq \pm 1, \pm 3 \\ = -\dfrac{27\omega_0^2}{c^2}\chi_{111}^{(2)} q_1^{(1)3}(\mathbf{U}\cdot\mathbf{U})\mathbf{U} \qquad \text{if } l = 3 \\ = \mathrm{i}\dfrac{\partial L_1}{\partial\omega_0}\dfrac{\partial\mathbf{E}_1^{(2)}}{\partial\tau} + \dfrac{1}{2}\dfrac{\partial^2 L_1}{\partial\omega_0^2}\dfrac{\partial^2\mathbf{E}_1^{(1)}}{\partial\tau^2} \\ \quad + \left(\mathrm{i}\dfrac{\partial q_1^{(1)}}{\partial\xi} - \dfrac{1}{2}\dfrac{\partial^2 k_0}{\partial\omega_0^2}\dfrac{\partial^2 q_1^{(1)}}{\partial\tau^2}\right)\left\{\dfrac{\partial}{\partial k_0}\left(L_1 - \dfrac{\omega_0^2 n_0^2}{c^2}\right)\right\}\mathbf{U} \\ \quad - |q_1^{(1)}|^2 q_1^{(1)}\dfrac{\omega_0^2}{c^2}\mathbf{F}(\mathbf{U}, \mathbf{U}^*; \chi^{(2)}) \qquad \text{if } l = 1 \end{cases} \tag{1.22}$$

where the column vector $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{U}, \mathbf{U}^*; \chi^{(2)}) = \chi_{-111}^{(2)}(\mathbf{U}^*\cdot\mathbf{U})\mathbf{U} + \chi_{1-11}^{(2)}(\mathbf{U}\cdot\mathbf{U}^*)\mathbf{U} + \chi_{11-1}^{(2)}(\mathbf{U}\cdot\mathbf{U})\mathbf{U}^*$$

Note if $\mathbf{U}$ is real and $\chi^{(2)}$ is symmetric, $\mathbf{F} = \chi^{(2)}(\mathbf{U}\cdot\mathbf{U})\mathbf{U}$.) From equation (1.22) one can obtain the solutions $\mathbf{E}_l^{(3)} = 0$ for $l \neq \pm 1$ or $\pm 3$, and since $L_3$ does not have an eigenmode,

$$\mathbf{E}_3^{(3)} = -\frac{27\omega_0^2}{c^2} q_1^{(1)3} L_3^{-1}\{\chi_{111}^{(2)}(\mathbf{U}\cdot\mathbf{U})\mathbf{U}\} \tag{1.23}$$

which is the harmonic generated by the non-linearity. For $l = 1$, we again require the compatibility condition

$$(\mathbf{U}, L_1\mathbf{E}_1^{(3)}) = 0 \tag{1.24}$$

from which we obtain the NLS equation for $q_1^{(1)}(\xi, \tau)$,

$$\mathrm{i}\frac{\partial q_1^{(1)}}{\partial\xi} - \frac{1}{2}\frac{\partial^2 k_0}{\partial\omega_0^2}\frac{\partial^2 q_1^{(1)}}{\partial\tau^2} + \nu|q_1^{(1)}|^2 q_1^{(1)} = 0 \tag{1.25}$$

Here, the Kerr coefficient $\nu$ is a positive real number given by

$$\nu = \frac{2\omega_0^2}{\tilde{k}_0 c^2 S_0}(\mathbf{U}, \mathbf{F}(\mathbf{U}, \mathbf{U}^*; \chi^{(2)})) \tag{1.26}$$

where $\tilde{k}_1 = k_1 - (2\mathrm{i}/S_0)\int(U_z\nabla_\perp\cdot\mathbf{U}^* - U_z^*\nabla_\perp\cdot\mathbf{U})\mathrm{d}S$. It is worth noting that the explicit form given in equation (1.21) for $\mathbf{E}_1^{(2)}$ is unnecessary in the calculation of the compatibility condition equation (1.24), and that equation (1.24) can be obtained directly from the equations for $\mathbf{E}_1^{(1)}$ and $\mathbf{E}_1^{(2)}$, i.e. equations (1.14) and (1.20).

In order to see the effect of the higher-order terms, one needs to find the equation for $q_1^{(2)}$ in (1.21). For this purpose, we have, at order $\varepsilon^4$, the expression $L_l\mathbf{E}_l^{(4)}$ for $l = 1$,