NONLINEAR MAGNETOHYDRODYNAMICS

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Introduction

Magnetohydrodynamics (MHD) describes the macroscopic behavior of electrically conducting fluids, notably of plasmas. However, in contrast to what the name seems to indicate, work in MHD has usually little to do with dynamics, or at least has had so in the past. In fact, most MHD studies of plasmas deal with magnetostatic configurations. This is not only a question of convenience – powerful mathematical methods have been developed in magnetostatic equilibrium theory – but is also based on fundamental properties of magnetized plasmas. While in hydrodynamics of nonconducting fluids static configurations are boringly simple and interesting phenomena are in general only caused by sufficiently rapid fluid motions, conducting fluids are often confined by strong magnetic fields for times which are long compared with typical flow decay times, so that the effects of fluid dynamics are weak, giving rise to quasi-static magnetic field configurations. Such configurations may appear in a bewildering variety of shapes generated by the particular boundary conditions, e.g. the external coils in laboratory experiments or the "foot point" flux distributions in the solar photosphere, and their study is both necessary and rewarding.

In addition to finding the appropriate equilibrium solutions one must also determine their stability properties, since in the real world only stable equilibria exist. Stability theory, however, often predicts instability for equilibrium solutions which appear to describe experimentally observed configurations quite well. What happens to these solutions if a weak perturbation is applied, do they merely relax into a neighboring equilibrium or slightly oscillating state, thus effectively enlarging the class of realizable equilibrium configurations? Such questions cannot be answered by linear stability theory.

A further aspect is connected with the various types of disruptive processes which are observed in laboratory and astrophysical plasmas to

occur occasionally after a period of quiescent plasma behavior. According to the conventional picture the configuration evolving because of slow changes of the boundary conditions becomes unstable at a certain point. A little reflection, however, shows that such an explanation is unsatisfactory and insufficient. Instabilities are usually weak for conditions close to the stability threshold, or marginal point, giving rise to a slow growth of the unstable perturbation, which completely misses the rapid explosive character of the observed process the instability is intended to explain. In addition, as mentioned above, linear instability theory does not allow an estimate of the final extent of the unstable dynamics. In particular, rapid linear growth rates do not guarantee that a large amount of energy is released. A somewhat more adequate approach to the problem of explosive processes appears to be equilibrium bifurcation theory. In particular a loss of equilibrium, called catastrophe, is often associated with the onset of rapid dynamics. However, a catastrophe usually occurs only within a certain equilibrium class, such that the system may still escape into a neighboring equilibrium state belonging to a more general class, for instance by introducing an X-type neutral point.

Hence it is necessary to leave the framework of equilibrium and stability theory and consider nonlinear dynamics explicitly. Mathematically this means leaving safe ground and embarking on unknown, perilous waters. In addition some price has in general to be paid for practical tractability. While equilibrium and stability theory can deal quantitatively with geometrically complicated systems, nonlinear MHD studies are usually restricted to obtain a qualitative picture in the simplest possible geometry. The pioneering papers date back to the early seventies, when theorists became aware of the importance of nonlinear effects. Among such papers are, notably, Rutherford's theory of the tearing mode evolution, Kadomtsev and Pogutse's theory of vacuum bubbles, the nonlinear theories of the ideal kink mode by Rosenbluth et al. and of the resistive kink mode by Kadomtsev, Syrovatskii's theory of current sheet formation and Taylor's theory of relaxed states, all of which have since been very influential. However, these nonlinear theories do not deal with truly dynamic processes but consider slowly evolving equilibrium states or asymptotic states of systems relaxed from some initial state under certain physical constraints. By contrast genuine dynamics is considered in a different line of approach, that of fully developed MHD turbulence. In the case of turbulence, nonlinear theory becomes tractable by applying statistical averaging together with some closure assumption. Here work started with Kraichnan's paper on the Alfvén effect in the sixties and a number of fundamental contributions, notably by Frisch, Montgomery and Pouquet, in the seventies.

For more general processes, however, numerical computations become the major tool, a trend observed in many branches of physics. In fact

the computational approach has reached a new dimension, which can be called computational theory. Fifty years ago a problem in fluid dynamics was considered as solved if the result could be expressed in terms of tabulated functions, twenty-five years ago if it could be reduced to an ordinary differential equation. With present-day supercomputers the partial differential equations for many two-dimensional fluid dynamic problems can be solved "exactly" for interesting Reynolds numbers and arbitrary boundary conditions. Moreover by performing a series of computer runs with different values of the externally given parameters scaling laws can be obtained. It appears that a problem should be considered solved if "exact" numerical solutions are available to provide scaling laws in a certain parameter regime, and when a basic physical mechanism is found, i.e. when the behavior of the system is "understood". This is the realm of computational theory. It might seem little in view of the beauty of exact analytical results; it should, however, be compared with the actually achievable, highly approximate analytical approaches often encountered. Since such approximations tend to be made more on grounds of convenience and feasibility than of mathematical rigor, they should be guided by the "exact" results obtained by relevant numerical simulation. Incidentally, the latter term, frequently used, is somewhat misleading. In fact one should distinguish between "physics simulations" and "real-world simulations". While the former simply provide an exact solution of some usually time-dependent model partial differential equations (exact within known and controllable discretization errors), the latter often incorporate many different, possibly equally important effects (e.g. in tokamak transport simulations), or complicated geometry (e.g. in stellerator development). In this book the term 'simulation' normally refers to the first category.

In a field such as nonlinear MHD, where no unifying methodical framework exists, the selection of topics is necessarily somewhat arbitrary, biased by personal taste. I have tried to concentrate more on the basic effects rather than include a broad scale of individual investigations. The book consists of three major parts, an introductory part, chapters 2–4, the main part treating three different aspects of nonlinear MHD, chapters 5–7, and an applications part, chapters 8–10.

Chapter 2 introduces the MHD equations in a macroscopic way, without recourse to concepts of kinetic theory. The ideal invariants play a crucial role, in particular magnetic helicity. We also derive a simplified set of equations for strongly magnetized plasmas, called reduced MHD, which has proved to be very convenient for nonlinear MHD studies. Finally the important dissipative effects are discussed, the magnitudes of which are measured by the corresponding Reynolds and Lundquist numbers.

In order to make the book self-contained, chapters 3 and 4 give an outline of the classical topics of MHD theory, equilibrium and stabil-

ity theory. Chapter 3 derives the general two-dimensional equilibrium equation, discusses some exact two-dimensional solutions often used as paradigms, and gives a brief overview of numerical methods to compute general equilibria in 2-D and 3-D.

The linear stability problem is addressed in chapter 4. Since the normal mode spectrum can analytically only be obtained for 1-D configurations, the energy principle has become the main tool for a qualitative stability analysis of more complicated systems. We first discuss the ideal MHD stability of a linear pinch and the modifications which arise in the toroidal case. Allowing for finite electrical resistivity significantly broadens the class of possibly unstable plasma motions. Two prototypes of resistive instabilities are treated in more detail, the tearing instability and the resistive kink instability, corresponding respectively to ideally stable and marginal unstable modes.

After these preliminaries the reader is prepared to enter the world of nonlinear processes. Throughout the book the emphasis is on relatively slow, essentially incompressible processes, excluding fast shock phenomena. Chapter 5 considers the laminar nonlinear evolution of MHD instabilities, where the system remains within the geometry defined by the most unstable mode, e.g. helical symmetry in the case of a kink mode in a cylindrical pinch. First the quasi-linear approximation is introduced, which provides a practical estimate of the instability saturation level in cases of a nonsingular final state. Then two rare examples of analytically solvable models are presented, the theory of vacuum bubbles resulting from external kink modes in an unsheared plasma cylinder, and the theory of the saturation of the ideal internal kink mode. These ideal MHD models are, however, of limited practical significance, since dynamical processes in magnetized plasma are strongly affected by the presence of finite resistivity even if the latter is numerically very small. Here the nonlinear tearing mode with its different variants is probably the most important individual MHD process, which is hence discussed in some detail, in particular the universal small-amplitude phase, called the Rutherford regime, and the saturation properties depending on the geometry of the configuration, the current distribution and the transport properties of the resistivity.

Magnetic reconnection, which in a stricter sense means the *fast* dynamic decoupling of plasma and magnetic field, can be called the essence of nonlinear MHD. Also in the case of ideal instability reconnection usually determines the nonlinear evolution. In magnetized plasmas reconnection takes place in current sheets. Chapter 6 first introduces the Sweet-Parker model, which incorporates the basic properties of dynamic current sheets. Reconnection theory has long been dominated by two schools of thought, Petschek's slow shock model and Syrovatskii's theory of current sheet generation. While the former has, however, been shown in recent years

to be invalid in the limit of high conductivity, for which it had been devised, the latter has been verified in detail by numerical simulations and in fact describes in a simple, elegant way a fundamental effect in the dynamics of highly conducting magnetized fluids. Though dynamic current sheets are significantly more stable than static ones they become tearing-unstable at sufficiently high Reynolds number, above which no stationary reconnection configurations appear to exist. As examples three well-known systems involving reconnection are discussed, the coalescence of magnetic islands, the nonlinear evolution of the resistive kink instability and the dynamics of plasmoids. While most of the chapter is restricted to two-dimensional systems, the final two sections discuss some aspects of three-dimensional reconnection.

Chapter 7 deals with fully developed turbulence, the most probable dynamical state at high Reynolds numbers. The absolute equilibrium distributions of truncated nondissipative systems provide insight into the properties of nonlinear mode interactions which determine the cascade directions in dissipative turbulence. In contrast to the hydrodynamic (Navier-Stokes) case MHD turbulence exhibits strong self-organization processes, which are connected with the existence of inverse cascades. These processes are selective decay leading to large-scale, static, force-free magnetic states, and dynamic alignment of velocity and magnetic fields. Also, spectral properties in MHD turbulence are different from those of hydrodynamic turbulence. The only quantitative approach for MHD turbulence theory developed to date is based on closure theory, a tractable version being the eddy-damped, quasi-normal Markovian approximation. Turbulent energy dissipation is discussed, in particular in 2-D MHD systems, which differs fundamentally from the behavior in 2-D hydrodynamics. An important aspect of modern turbulence theory is intermittency. Several intermittency models developed for Navier-Stokes turbulence are introduced and some results for MHD turbulence are given. Finally the magnetoconvection in primarily unmagnetized fluid turbulence is discussed, which is intimately connected with the turbulent dynamo effect.

The remaining chapters are devoted to three important applications in laboratory and astrophysical plasmas. Chapter 8 discusses the MHD properties of disruptive processes observed in tokamaks. The sawtooth oscillation is a periodic relaxation process restricted to the central region of the plasma column. Observations show beyond reasonable doubt that the sawtooth collapse is connected with the $m = 1, n = 1$ kink mode. However, Kadomtsev's model assuming full reconnection of the helical magnetic flux by the resistive kink mode, which had long been the generally accepted sawtooth model, does not seem to apply to present-day large-diameter hot tokamak plasmas, characterized by very large values

of the Lundquist number S. Observed time scales seem to be too fast to allow full reconnection. In fact, measurements of the central safety factor indicate that full reconnection does not take place and that the thermal energy release is caused by some effect which is different from the convective process in Kadomtsev's model. Numerical simulations, being still confined to rather low S-values, have not been able to elucidate the high S-value behavior. In addition the fast onset of the collapse is still poorly understood. The major disruption in a tokamak occurs accidentally, when plasma parameters, in particular current and pressure, exceed certain operational limits, the density limit being effectively a current limit due to transport processes. The disruption proper appears to be a fast MHD process caused by the nonlinear instability of a large amplitude $m = 2$ tearing mode, which leads to a turbulent state. It can be described as an interaction of modes of different helicity, notably $(m, n) = (2, 1), (1, 1), (3, 2)$, which gives rise to an anomalous resistivity. The third type of disruptive process is again a quasi-periodic relaxation oscillation, affecting the outer plasma region, hence the name 'edge-localized mode' (ELM). It occurs primarily in divertor plasmas in the high-confinement (H-) regime, where owing to a local transport barrier steep pressure gradients are generated at the plasma edge. The ELM can be associated with high-m ballooning modes. A common feature observed in all three types of disruptive events is that of a two-stage process, consisting of a coherent precursor and a more rapid turbulent relaxation phase.

The reversed-field pinch (RFP) considered in chapter 9 is particularly rich in MHD effects. Because of the low value of the safety factor the RFP is prone to instability in contrast to a tokamak plasma and hence tends to relax to the minimum energy configuration with a reversed toroidal field predicted by Taylor's theory, which we discuss in some detail. RFP plasmas are usually maintained in a quasi-stationary state, where turbulent relaxation to the minimum energy state is counteracted by resistive diffusion. The process of transforming the externally supplied poloidal field into internal toroidal field is called the RFP dynamo effect. The simplest theoretical model is a helical ohmic state, where a helical stationary flow balances resistive diffusion. Such stationary dynamo states are not forbidden by Cowling's antidynamo theorem, which applies only to the more restricted case of axisymmetry. In general, however, such stationary states are unstable with respect to modes with different helicities. The resulting turbulent behavior can only be investigated by numerical simulation.

Finally, chapter 10 deals with solar flares, which are among the most spectacular explosive events observed in astrophysical plasmas. Flares result from a sudden release of magnetic energy and are hence MHD processes, though a treatment of the different channels of energy dissipation

which give rise to the wealth of observed phenomena is outside the scope of this book. We first discuss the process of magnetic field generation in the solar convection zone and the typical magnetic configurations emerging into the corona. Flares appear over a wide energy range, from the very weak microflares or even nanoflares forming a background noise responsible for the continuous coronal heating, to the macroflares occuring sporadically. The latter are loosely classified as simple loop or compact flares and two-ribbon flares, the largest events. The current MHD models are presented for flares of both types.

A few remarks concerning the notation used in the book may be in place. I have tried as much as possible to stay within the notations used in the current literature. Occasionally conflicts arise, for instance for the energy which is usually denoted by W in stability theory, but by E in turbulence theory. I follow such conventions, but give an explicit warning. The equations are written in SI units, which seem to be the most practical for macroscopic plasma physics. However, to simplify notation I will set the vacuum permeability $\mu_0 = 1$. If necessary μ_0 can be reintroduced at any stage by a simple dimensionality consideration. In addition, since most of the MHD processes considered are incompressible, a homogeneous density distribution ρ is often assumed, setting $\rho = 1$. In these units the magnetic field B has the dimension of a velocity, the corresponding Alfvén velocity $v_A = B/\sqrt{\mu_0 \rho}$, and v_A and B will be used interchangeably.