COMPLEX POLYNOMIALS

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1.1 Complex polynomials

1.1.1 Definitions

A **complex polynomial** is a function of the form

$$P(z) = \sum_{k=0}^{n} a_k z^k, \tag{1.1}$$

where the a_k are complex numbers not all zero and where z is a complex variable. We also use the terms **analytic** polynomial (reflecting the fact that the polynomial is an analytic function) and **algebraic** polynomial (since the polynomial contains only algebraic operations on the variable z). If $a_n \neq 0$ the polynomial is said to have **degree** n. In particular, a polynomial of degree 0 is, by definition, a non-zero constant. The function which is identically zero is often regarded as being a polynomial of degree $-\infty$. When the a_k are all real numbers, the polynomial $P(z)$ is called a **real polynomial**. Observe that $P(z)$ is a real polynomial iff

$$\overline{P(\overline{z})} = P(z) \tag{1.2}$$

for all $z \in \mathbb{C}$. From this it follows that, if $P(a) = 0$ then $P(\overline{a}) = 0$; therefore either a is real or P has the **conjugate pair** of zeros a and $\overline{a}$.

1.1.2 Number of zeros

Lemma *A complex polynomial of degree n has* ***at most*** *n zeros.*

The proof of this is an entirely elementary fact from algebra; this is to be contrasted with the stronger **fundamental theorem of algebra** (see chapter 2) which states that a polynomial of degree $n \geq 0$ has exactly n zeros; the usual

proofs of this use methods of analysis or topology (it is not a result which follows purely from the algebraic field property of the complex numbers). The proof of the weaker statement is by induction. The result is trivial when $n = 0$. Assume the statement proved for polynomials of degree $n-1$, where $n \geq 1$, and let P be a polynomial of degree n given say by (1.1). Either P has no zeros and there is nothing further to prove, or $\exists a \in \mathbb{C}$ such that $P(a) = 0$. We then have

$$\frac{P(z)}{z-a} = \frac{P(z)-P(a)}{z-a} = \sum_{k=0}^{n} a_k \frac{z^k - a^k}{z-a} = \sum_{k=1}^{n} a_k \sum_{j=0}^{k-1} z^{k-1-j} a^j, \quad (1.3)$$

and the last expression is clearly a polynomial of degree $n-1$. Therefore by the induction hypothesis $P(z)/(z-a)$ has at most $n-1$ zeros and so $P(z)$ has at most n zeros. The result follows by induction.

This result has a number of important consequences.

1.1.3

Uniqueness theorem *If $P(z)$ and $Q(z)$ are polynomials of degree not exceeding n and if the equation*

$$P(z) = Q(z) \quad (1.4)$$

is satisfied at $n+1$ distinct points, then $P = Q$.

For otherwise, $P - Q$ is a polynomial of degree not exceeding n with $n+1$ zeros. We deduce the next result.

1.1.4

Theorem: Lagrange's interpolation formula *Let $z_1, z_2, \ldots, z_{n+1}$ be $n+1$ distinct points and let $w_1, w_2, \ldots, w_{n+1}$ be arbitrary complex numbers (not necessarily distinct but not all zero). Among all polynomials of degree not exceeding n there is a unique polynomial $P(z)$ such that*

$$P(z_k) = w_k \quad (1 \leq k \leq n+1). \quad (1.5)$$

This has the representation

$$P(z) = \sum_{k=1}^{n+1} w_k \frac{Q(z)}{Q'(z_k)(z-z_k)}, \quad (1.6)$$

where $Q(z) = \prod_{k=1}^{n+1}(z-z_k)$.

By the previous result there is at most one such polynomial; on the other hand the polynomial so represented is immediately seen to have the desired property.

1.1.5

An alternative formulation of Lagrange's formula is the following.

Corollary *Let $z_1, z_2, \ldots, z_{n+1}$ be $n+1$ distinct points and let $Q(z) = \prod_{k=1}^{n+1}(z - z_k)$. If $P(z)$ is a polynomial of degree not exceeding n, then*

$$P(z) = \sum_{k=1}^{n+1} P(z_k)\frac{Q(z)}{Q'(z_k)(z - z_k)}. \tag{1.7}$$

In other words we have an explicit method for determining the values of a polynomial of degree n in terms of its values at $n+1$ known points.

1.1.6

Example Let $Q(z) = z^{n+1} - 1 = \prod_{k=0}^{n}(z - \omega_k)$, where $\omega_k = e^{2\pi ik/(n+1)}$ are the $(n+1)$th roots of unity. Then, if $P(z)$ is a polynomial of degree not exceeding n, we have

$$P(z) = \frac{1}{n+1}\sum_{k=0}^{n} P(\omega_k)e_n(\overline{\omega}_k z), \tag{1.8}$$

where $e_n(z) = 1 + z + \cdots + z^n$.

1.1.7 Representation for harmonic polynomials

A **harmonic polynomial** $T(z)$ is a function of the form $T(z) = \overline{Q(z)} + P(z)$, where Q and P are analytic polynomials, and so is a complex-valued harmonic function in $\mathbb{C}$ (the complex plane). T can be represented in the form

$$T(z) = \sum_{k=-n}^{n} a_k r^{|k|} e^{ik\theta} \quad (z = re^{i\theta}). \tag{1.9}$$

T has **degree** n if either a_n or a_{-n} is non-zero. Note that a harmonic polynomial is the sum of a polynomial in the variable $\overline{z}$ and a polynomial in the variable z. The polynomial is uniquely determined by the $2n+1$ coefficients a_k. These coefficients also determine uniquely the **trigonometric polynomial**

$$T(e^{i\theta}) = \sum_{k=-n}^{n} a_k e^{ik\theta}. \tag{1.10}$$

Indeed, given this latter expression the coefficients can be recovered from the formula

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}T(e^{i\theta})d\theta \quad (-n \le k \le n). \tag{1.11}$$

We can apply Lagrange's interpolation formula to obtain a representation for a harmonic polynomial as follows. Let $\zeta_k = e^{2\pi ik/(2n+1)}$ $(0 \le k \le 2n)$ and set

$$D_n(z) = \overline{z}^n + \cdots + \overline{z} + 1 + z + \cdots + z^n. \tag{1.12}$$

Then, *if $T(z)$ is a harmonic polynomial of degree not exceeding n, we have*

$$T(z) = \frac{1}{2n+1}\sum_{k=0}^{2n} T(\zeta_k)D_n(\overline{\zeta}_k z). \tag{1.13}$$

This follows easily by applying example 1.1.6 to $e^{in\theta}T(e^{i\theta})$, which is a polynomial of degree at most $2n$ in the variable $e^{i\theta}$. We obtain the above formula on the unit circle. Since both sides of the equation are harmonic polynomials, it follows that the equality holds for all $z \in \mathbb{C}$.

1.2 The number of zeros of a real analytic polynomial

1.2.1

Definition A **real analytic** polynomial is an expression of the form

$$P(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{j,k}x^j y^k \tag{1.14}$$

where the coefficients $a_{j,k}$ are real or complex numbers and where x and y are real variables. The **degree** of the term $a_{j,k}x^j y^k$ is $j + k$ provided that $a_{j,k} \neq 0$, and the **degree** of P is the largest of the degrees of the individual terms. Every algebraic polynomial and every harmonic polynomial is real analytic and their degrees earlier defined agree with the above definition.

1.2.2

Bézout's theorem *Let $P(x, y) = u(x, y) + iv(x, y)$ (with u and v real) be a complex-valued real analytic polynomial, where u has degree m and v has degree n, and suppose that u and v are relatively prime (i.e. contain no non-trivial common factors). Then P has at most mn zeros in $\mathbb{C}$.*

Proof This is an algebraic result which is proved using linear techniques. Firstly, we may note that $P(0, y)$ is not identically zero, for otherwise x is

a common factor of u and v. We then write u and v in the form

$$u(x,y)=\sum_{k=0}^{m}u_k(x)y^k,\qquad v(x,y)=\sum_{k=0}^{n}v_k(x)y^k, \tag{1.15}$$

where u_k are real polynomials in x of degree at most $m-k$ and v_k are real polynomials in x of degree at most $n-k$. Also, after an affine transformation of the variables, we may assume that u_m and $v_n \neq 0$. (**Affine transformations** are first degree real analytic polynomial mappings, which are bijective mappings of $\mathbb{C}$; for an account of these see chapter 2, section 2.7.21.) Now the equation $P=0$ holds iff $u=0$ and $v=0$ simultaneously. Let us assume this is the case at a particular pair (x,y). We use the **method of Sylvester** to eliminate the quantity y from the two equations $u(x,y)=0$, $v(x,y)=0$. The method is to observe that the following $m+n$ equations hold:

$$y^{\mu}u(x,y)=0\quad(0\le\mu\le n-1);\qquad y^{\nu}v(x,y)=0\quad(0\le\nu\le m-1). \tag{1.16}$$

Here $y^0=1$ by definition. Each of these equations can be interpreted as the vanishing of a linear combination of the $m+n$ quantities $1, y, y^2, \ldots, y^{m+n-1}$. This implies the vanishing of the determinant of coefficients:

$$D(x)=\begin{vmatrix} u_0(x) & u_1(x) & u_2(x) & 0 \\ 0 & u_0(x) & u_1(x) & u_2(x) \\ v_0(x) & v_1(x) & v_2(x) & 0 \\ 0 & v_0(x) & v_1(x) & v_2(x) \end{vmatrix}=0. \tag{1.17}$$

This example is the case $m=n=2$. The determinant $D(x)$ is a polynomial in x. We will show that (a) $D(x)$ *does not vanish identically* and (b) $D(x)$ *has degree at most* mn. When these facts are established, it will follow that there are at most mn values of x for which the determinant can vanish. For each such x there are only finitely many values of y such that $P(x,y)=0$. Thus we obtain at most finitely many points (x,y) for which $P(x,y)=0$. Because of this we may now perform an affine transformation on (x,y) so that, after the transformation to new variables x', y', the resulting polynomial has for each fixed x' at most one zero in y'. (This is easily seen geometrically; e.g. a small rotation will do.) It follows immediately from the above applied to the new coordinates that P has at most mn zeros altogether, proving the theorem.

To prove (b) we let $w_{i,j}$ denote the terms of the $(m+n)\times(m+n)$ matrix of coefficients and observe that each term is a polynomial in x whose degree satisfies

$$\text{degree of } w_{i,j}\le\begin{cases} m-j+i & (1\le i\le n), \\ -j+i & (n+1\le i\le m+n). \end{cases} \tag{1.18}$$

Here we have taken the degree of the identically zero polynomial to be $-\infty$. Now the value of the determinant is given by the formula

$$|w_{i,j}| = \sum (-1)^{\sigma} \prod_{i=1}^{m+n} w_{i,\phi(i)} \tag{1.19}$$

where the sum ranges over all permutations $\phi(i)$ of the numbers $1, 2, \ldots, m+n$, and where $\sigma = 0$ for even permutations and $\sigma = 1$ for odd permutations. It follows that the degree of the determinant cannot exceed

$$\sum_{i=1}^{n}(m - \phi(i) + i) + \sum_{i=n+1}^{m+n}(-\phi(i) + i) = mn + \sum_{i=1}^{m+n}(-\phi(i) + i) = mn, \tag{1.20}$$

since $\sum \phi(i) = \sum i$, as $\phi(i)$ runs through the numbers i in some order. This proves (b).

To prove (a) we assume that the determinant vanishes identically and show that this implies that u and v have a common factor, contradicting the hypotheses. Firstly, we observe that the vanishing of the determinant is equivalent to the vanishing of the determinant of the transpose matrix. Secondly, we observe that the elements of the matrix are polynomials in x and therefore belong to the field of rational functions in x. Because the determinant vanishes for all x, we may interpret this as the statement that the determinant of this matrix consisting of elements of this field vanishes, as all the operations required to evaluate the determinant are algebraically defined over any field. Therefore from the algebraic theory of determinants we can assert the existence of elements α_k, β_k of the field not all zero and satisfying the set of equations

$$u_0\alpha_0 + v_0\beta_0 = 0, \qquad u_1\alpha_0 + u_0\alpha_1 + v_1\beta_0 + v_0\beta_1 = 0, \qquad \ldots \text{etc.} \tag{1.21}$$

or generally

$$\sum_{k=r-m}^{r} u_{r-k}\alpha_k + \sum_{k=r-n}^{r} v_{r-k}\beta_k = 0 \quad (0 \le r \le m+n-1), \tag{1.22}$$

where $\alpha_k = 0$ for $k \ge n$ and $\beta_k = 0$ for $k \ge m$. We now set

$$A(y) = \sum_{k=0}^{n-1} \alpha_k y^k, \qquad B(y) = -\sum_{k=0}^{m-1} \beta_k y^k \tag{1.23}$$

and note that by multiplying out and equating coefficients to zero the above set of equations is equivalent to the single equation

$$uA - vB = 0. \tag{1.24}$$

Since not both A and B are zero, it follows from this equation that neither A nor B is the zero element, as otherwise one of u and v is zero, contradicting the hypothesis. Now the coefficients of A and B are rational functions in x; therefore multiplying through by a suitable polynomial in x, we obtain an equation of the same form with A and B polynomials in both x and y. Furthermore, we may divide out the equation by any common factor of A and B. Thus we obtain an equation of the above form with A and B relatively prime polynomials, where the degree of A in y is smaller than the degree of v in y and the degree of B in y is smaller than the degree of u in y. We will show that $f = v/A = u/B$ is a polynomial in x and y, which is then the required common factor of u and v. Consider first the case when B is constant in the variable y, and so a pure polynomial in x. B can be factorised as a product of irreducible polynomials; hence it is enough to show that each irreducible factor R of B is a factor of u. Expanding u in powers of y, this is true iff every coefficient of u is divisible by R. Write $u = u_1 + u_2$, where u_1 consists of those terms in the expansion of u whose coefficients are divisible by R, and u_2 consists of the remaining terms. Similarly, write $A = A_1 + A_2$. Since R is not a factor of A, $A_2 \neq 0$. Thus we have $vB = Au = A_1u_1 + A_1u_2 + A_2u_1 + A_2u_2$, from which we see that A_2u_2 is divisible by R. This implies that $u_2 = 0$, as otherwise the lowest coefficient of this product in y is divisible by R, but also is the product of a coefficient of A_2 and a coefficient of u_2, which is not divisible by R. Thus B divides u in this case.

Returning to the general case, from the usual division algorithm for polynomials we can write

$$\frac{v}{A} = f + \frac{\rho}{A}, \qquad \frac{u}{B} = g + \frac{\sigma}{B} \tag{1.25}$$

where f, g, ρ and σ are polynomials in y with rational coefficients in x, and where ρ has y-degree smaller than A and σ has y-degree smaller than B. Hence as $y \to \infty$, ρ/A and $\sigma/B \to 0$ and so $f - g \to 0$. This clearly implies that $f = g$. We therefore obtain $\rho/A = \sigma/B$ and so either (i) $\rho = \sigma = 0$ or (ii) $A/\rho = B/\sigma$. In case (ii) we see that we are back to the same problem for A and B that we had for u and v, but A and B have smaller degrees. Therefore we may use an induction argument to deduce that A and B have a common factor, contradicting our legitimate assumption. It follows that case (i) holds and so f is a common factor of u and v. The coefficients of f are rational functions in x, but by multiplying through by the smallest polynomial p in x to cancel out the denominators of the coefficients, we find as above that p divides both A and B. Since A and B are relatively prime, f is a polynomial in both x and y. This proves (a) and so completes the proof of Bézout's theorem.

1.2.3

Bézout's theorem can be used to put a bound on the number of zeros of a real analytic polynomial even in the absence of algebraic information on the polynomial. For example it may often be the case that on analytic or topological grounds one can establish the finiteness of the number of zeros. In this case we have the following result.

Theorem *Let $P(x, y) = u(x, y) + i\,v(x, y)$ be a complex-valued real analytic polynomial, where u has degree $m \geq 0$ and v has degree $n \geq 0$. Then P has at most mn isolated zeros in $\mathbb{C}$. In particular, if P has at most finitely many zeros, then P has at most mn zeros in $\mathbb{C}$.*

Proof The following argument is an adaptation of a method given by Wilmshurst [72]. If $m = 0$, then u is a non-zero constant and so P has no zeros; similarly if $n = 0$. If $m = 1$, then either (i) u divides v or (ii) u and v are relatively prime. In case (ii) Bézout's theorem shows that P has at most n zeros; in case (i) P vanishes exactly when u vanishes, which is on a line. Hence P has no isolated zeros. Thus the theorem follows in this case. We proceed by induction and assume the theorem is true for polynomials $U + iV$, where U has degree smaller than m and V has degree smaller than n. If u and v are relatively prime, then Bézout's theorem gives the result. Therefore we may assume that u and v have highest common factor p say, where p is a real polynomial in (x, y) of degree $r \geq 1$. Then u/p and v/p are relatively prime polynomials of degrees $m - r$ and $n - r$ respectively, and so P/p has at most $(m - r)(n - r)$ zeros. The remaining zeros of P are the zeros of p. Now p is a real-valued continuously differentiable function in the plane. It follows from the implicit function theorem (see chapter 2) that at a zero c of p either (i) p vanishes on a curve passing through c or (ii) both partial derivatives p_x and p_y are zero at c. In case (i) c is a non-isolated zero of P. Thus if c is an isolated zero of P, which is also a zero of p, then c is an isolated zero of p, and so from case (ii) the polynomial $Q = p + ip_x$ has an isolated zero at c. Since p has degree r and p_x has degree $r - 1$, it follows by the induction hypothesis that Q has at most $r(r - 1)$ isolated zeros. Hence P has at most $(m - r)(n - r) + r(r - 1) < mn$ isolated zeros. The result follows by induction.

1.2.4 A complex approach

It is clear that Bézout's theorem will be equally valid for a pair of complex polynomials $u(z, \zeta)$, $v(z, \zeta)$ of complex variables z and ζ. From the two equations $u(z, \zeta) = 0$ and $v(z, \zeta) = 0$ we can use Sylvester's method to eliminate

the variable ζ, obtaining a determinant $D(z)$, which is an algebraic polynomial in z whose zeros contain those z for which there exists a pair (z, ζ) satisfying the two equations. The maximum degree of D is the product of the degrees of u and v. We can apply this result to our original problem of finding the zeros of a complex-valued real analytic polynomial P. P is a polynomial in the real variables x and y; by making the substitutions $x = \frac{1}{2}(z + \overline{z})$, $y = \frac{1}{2i}(z - \overline{z})$, P becomes a polynomial $P(z, \overline{z})$ in the 'variables' z and $\overline{z}$; in other words a real analytic polynomial can be written in the form of a finite sum

$$P(z, \overline{z}) = \sum_{j,k} a_{j,k} \overline{z}^j z^k, \tag{1.26}$$

where the $a_{j,k}$ are complex numbers and the degree of P is the maximum of $j + k$ over terms where $a_{j,k} \neq 0$. This is clearly the restriction of $P(z, \zeta) = \sum_{j,k} a_{j,k} \zeta^j z^k$ to $\zeta = \overline{z}$. To apply the Sylvester method of elimination we require two equations. However, $P(z, \overline{z}) = 0$ iff $\overline{P(z, \overline{z})} = 0$; thus we eliminate ζ from the two equations

$$P(z, \zeta) = 0, \qquad \overline{P(\overline{\zeta}, \overline{z})} = 0 \tag{1.27}$$

which take the form

$$\sum_{j,k} a_{j,k} \zeta^j z^k = 0, \qquad \sum_{j,k} \overline{a_{j,k}} z^j \zeta^k = 0. \tag{1.28}$$

We obtain the determinant equation $D(z) = 0$, where D is an algebraic polynomial in z whose degree does not exceed n^2, where n is the degree of P. D is identically zero iff the above pair of polynomials in z and ζ have a common factor. We leave it as an exercise to show that this holds iff, writing $P = u + iv$ (u, v real), u and v have a common factor. Thus in this formulation we have lost the distinction between the possible different degrees of u and v, as n is clearly the larger of these two degrees. On the other hand we have constructed an algebraic polynomial $D(z)$ whose set of zeros contains all the zeros of the original real analytic polynomial P. In fact, if (z, ζ) is any pair satisfying the above two equations, then $D(z) = 0$. In general it need not be the case that the ζ satisfies $\zeta = \overline{z}$, and therefore D is likely to have more zeros than P. However, there are certainly cases where D and P have the same zeros. Indeed this is the case if P is itself an algebraic polynomial: then $D = cP^n$, where c is a constant $\neq 0$. Of course, if we count multiplicities, then D has n^2 zeros, whereas P has only n zeros. We will give an example in chapter 2 (subsection 2.6.11) where D has n^2 simple zeros all of which are zeros of P.

It is worth recording here the form of the $2n \times 2n$ matrix of which D is the determinant. We have

$$P(z,\overline{z}) = \sum_{j,k} a_{j,k}\overline{z}^j z^k = \sum_j \left(\sum_k a_{j,k} z^k \right) \overline{z}^j = \sum_k \left(\sum_j a_{j,k} \overline{z}^j \right) z^k \quad (1.29)$$

and so

$$P(z,\overline{z}) = \sum_{j=0}^{n} b_j(z)\overline{z}^j, \qquad \overline{P(z,\overline{z})} = \sum_{j=0}^{n} c_j(z)\overline{z}^j, \quad (1.30)$$

where

$$b_j(z) = \sum_{k=0}^{n-j} a_{j,k} z^k, \qquad c_j(z) = \sum_{k=0}^{n-j} \overline{a_{k,j}} z^k. \quad (1.31)$$

The terms of the matrix are given by

$$\begin{aligned} w_{i,j} &= b_{j-i} \quad \text{for } i \le j \le n+i \quad \text{and} \quad 1 \le i \le n, \\ w_{i,j} &= c_{j-i+n} \quad \text{for } i-n \le j \le i \quad \text{and} \quad n+1 \le i \le 2n, \\ w_{i,j} &= 0 \quad \text{otherwise.} \end{aligned} \quad (1.32)$$

As recorded earlier these relations imply that the maximum degree of $w_{i,j}$ is given by

$$\text{degree of } w_{i,j} \le \begin{cases} n-j+i & (i \le j \le n+i \text{ and } 1 \le i \le n), \\ -j+i & (i-n \le j \le i \text{ and } n+1 \le i \le 2n). \end{cases} \quad (1.33)$$

Otherwise degree of $w_{i,j} = -\infty$ (so trivially satisfies the above inequalities). We then have

$$D(z) = \sum (-1)^\sigma \prod_{i=1}^{2n} w_{i,\phi(i)} \quad (1.34)$$

where the sum ranges over all permutations $\phi(i)$ of the numbers $1, 2, \ldots, 2n$, and where $\sigma = 0$ for even permutations and $\sigma = 1$ for odd permutations. Each product $\prod_{i=1}^{2n} w_{i,\phi(i)}$ has degree not exceeding $\sum_{i=1}^{n}(n - \phi(i) + i) + \sum_{i=n+1}^{2n}(-\phi(i)+i) = n^2$, with equality only if every term $w_{i,\phi(i)}$ in the product attains its maximum degree $n - \phi(i) + i$ for $1 \le i \le n$ or $-\phi(i) + i$ for $n+1 \le i \le 2n$.

As an application of these remarks, suppose that P has coefficients $a_{j,k}$ where $a_{j,n-j} = 0$ for $1 \le j \le n$ and $a_{0,n} \ne 0$; in other words the only term of highest degree n in the expansion of P is the term in z^n. Then $b_0(z)$ has full degree n, and $c_0(z)$ has degree 0, but the remaining polynomials $b_j(z)$ $(1 \le j \le n)$ and $c_j(z)$ $(0 \le j \le n-1)$ have degree strictly smaller than $n - j$. It follows that the

only permutation giving a product of degree n^2 is the case $\phi(i) = i$ $(1 \leq i \leq 2n)$. Thus $D(z)$ has exact degree n^2, and therefore P has at most n^2 zeros.

Collecting up the results of this subsection we have the next theorem.

1.2.5

Theorem *Let $P(z, \overline{z}) = u + iv$ (u and v real) be a complex-valued real analytic polynomial of degree n. Then the Sylvester resultant $D(z)$ of $P(z, \zeta)$ and $\overline{P(\overline{\zeta}, \overline{z})}$ is a polynomial in z of degree at most n^2. Every zero of $P(z, \overline{z})$ is a zero of $D(z)$; indeed, if $P(z, \zeta) = \overline{P(\overline{\zeta}, \overline{z})} = 0$, then $D(z) = 0$. $D(z)$ vanishes identically if, and only if, u and v have a non-trivial common factor. Finally, if*

$$P(z, \overline{z}) = Q(z, \overline{z}) + Az^n, \tag{1.35}$$

where Q is a real analytic polynomial of degree $< n$ and A is a non-zero constant, then $D(z)$ has exact degree n^2 and therefore P has at most n^2 zeros. Indeed, in this case, P attains every value w at most n^2 times and is therefore an n^2-valent mapping of the plane.

The final remark comes from applying the theorem to $P - w$. We shall see in chapter 2 that P of this form does actually attain every value w, and so is a surjective mapping of the plane.

1.2.6 The Sylvester resultant of $P - w$

In studying the valence of a mapping $P(z, \overline{z})$ it is natural to consider the Sylvester resultant for $P - w$, where w is an arbitrary complex number. The resultant will take the form $D(z, w, \overline{w})$, where this is an algebraic polynomial in z and a real analytic polynomial in w. Furthermore, if P has degree n, then as a real analytic polynomial in w, D has at most degree n. This follows from the fact that the only terms involving w in the matrix are the terms $b_0 - w$ and $c_0 - \overline{w}$, so w only appears with degree 1 in $w_{i,j}$ if $j = i$ for $1 \leq i \leq n$ and if $j = i - n$ for $n + 1 \leq i \leq 2n$. All other terms have degree 0 or $-\infty$ in w. For any permutation ϕ of the numbers $1, 2, \ldots, 2n$, there are at most n values i such that (i) $\phi(i) = i$ where $1 \leq i \leq n$ and (ii) $\phi(i) = i - n$ where $n + 1 \leq i \leq 2n$.

1.2.7 Evaluating the Sylvester resultant

Let $b_0, b_1, \ldots, b_n$ and $c_0, c_1, \ldots, c_n$ be the generating elements of the $2n \times 2n$ Sylvester matrix. Then the resultant can be written in the form

$$\sum_{\phi} (-1)^{\sigma} \prod_{i=1}^{n} b_{\phi(i)-i}\, c_{\phi(i+n)-i} \tag{1.36}$$

summed over all permutations ϕ of the numbers $1, 2, \ldots, 2n$ such that for $1 \le i \le n$

$$0 \le \phi(i) - i \le n, \qquad 0 \le \phi(i+n) - i \le n. \tag{1.37}$$

Note that, for each permutation ϕ, the sum of the subscripts of the bs and the cs is exactly n^2.

1.2.8 Further properties of the resultant

Let u and v be real analytic polynomials of degrees m and n respectively written in the form

$$u(x,y) = \sum_{k=0}^{m} u_k(x) y^k, \qquad v(x,y) = \sum_{k=0}^{n} v_k(x) y^k, \tag{1.38}$$

where $u_m \neq 0$ and $v_n \neq 0$. If the resultant is not identically zero, then the set of equations

$$\sum_{k=r-m}^{r} u_{r-k}\alpha_k + \sum_{k=r-n}^{r} v_{r-k}\beta_k = \lambda_r \quad (0 \le r \le m+n-1) \tag{1.39}$$

can be uniquely solved for α_k and β_k, where $\alpha_k = 0$ for $k \ge n$ and $\beta_k = 0$ for $k \ge m$. Here the λ_r are given arbitrary elements in the field of rational functions in x, and the solutions α_k and β_k are then also elements in this field. Writing

$$\Lambda(x,y) = \sum_{r=0}^{m+n-1} \lambda_r(x) y^r, \quad A(x,y) = \sum_{k=0}^{n-1} \alpha_k(x) y^k, \quad B(x,y) = \sum_{k=0}^{m-1} \beta_k(x) y^k, \tag{1.40}$$

the above equations are equivalent to the single equation

$$uA + vB = \Lambda. \tag{1.41}$$

In other words, *if the resultant of u and v does not vanish identically, then given Λ of degree in y at most $m+n-1$ we can find a unique A of degree at most $n-1$ in y and a unique B of degree at most $m-1$ in y to satisfy this equation.*

In particular, taking $\Lambda = 1$, *we can find a unique p of degree at most $n-1$ and a unique q of degree at most $m-1$ such that*

$$up + vq = 1. \tag{1.42}$$

Here p and q are polynomials in y with coefficients which are rational functions in x. If $R(x)$ is the lowest common denominator of these rational coefficients, then we obtain the relation

$$uP + vQ = R \tag{1.43}$$

where $P = pR$, $Q = qR$ are now polynomials in both x and y. The polynomial $R(x)$ is a factor of the resultant: if we use the Sylvester matrix to solve the equation $up + vq = 1$ for p and q, then, applying Cramer's rule, the solutions for each of the coefficients of p and q (expanded as polynomials in y) will be the ratio of two determinants; the numerator is $\pm$ the determinant of one of the minors of the matrix corresponding to the first column; the denominator is the determinant of the matrix, i.e. the Sylvester resultant. Since u and v have polynomial coefficients, it follows that the Sylvester resultant is a common denominator of all the rational coefficients of p and q. Hence the lowest common denominator R is a factor of the resultant; if R is a proper factor, then there is a factor of the resultant which is a common factor of the determinants of each of the minors corresponding to the first column.

1.3 Real analytic polynomials at infinity

1.3.1 Resolving the singularity: the blow-up method

Let $P(x, y)$ be a real analytic polynomial of degree N. The limiting behaviour at infinity of P can be determined by performing a sequence of transformations, which we now describe. Firstly, after a suitable affine transformation we may assume that both the real and imaginary parts of $P(0, y)$ have full degree N. We then make the transformation

$$x \mapsto 1/x,\ y \mapsto y/x, \tag{1.44}$$

obtaining

$$R(x, y) = P\left(\frac{1}{x}, \frac{y}{x}\right) = \frac{Q(x, y)}{x^N}, \tag{1.45}$$

where $Q(x, y)$ is a polynomial of degree N such that $Q(0, y)$ has degree N. Any finite limiting value of P at infinity is then a limiting value of R as $x \to 0$ and y remains bounded. We expand Q in powers of x, obtaining

$$R(x, y) = \frac{\sum_{k=0}^{N} Q_k(y)x^k}{x^N}. \tag{1.46}$$

It is clear that $R(x, y) \to \infty$ as $x \to 0$ and y remains bounded, unless y tends to a real zero of $Q_0(y)$; for otherwise the term $Q_0(y)/x^N$ dominates the expression. Thus to determine the possible finite limiting values of R we will consider $x \to 0$ through positive values and $y \to a$, where a is a real zero of Q_0. We write

$$Q_k(y) = (y - a)^{r_k} S_k(y) \quad (0 \le k \le N), \tag{1.47}$$

where r_k is the multiplicity of the zero a of Q_k (if $Q_k(a) \neq 0$, then $r_k = 0$); if $Q_k \equiv 0$, for the purposes of this discussion we may take $r_k = +\infty$. Next, let us consider the substitution

$$y = a + tx^p, \tag{1.48}$$

where p is a positive number to be determined. Note that the mapping $(x, y) \mapsto (x, t)$ is a 1–1 mapping of the half-plane $x > 0$ onto itself. We obtain

$$R(x, y) = x^{-N}\left(\sum_{k=0}^{N} S_k(y) t^{r_k} x^{pr_k+k}\right). \tag{1.49}$$

Now observe that, for small positive p, $pr_0 < pr_k + k\,(1 \leq k \leq N)$ and the dominating term is $S_0(a)t^{r_0}/x^{N-pr_0}$. If $pr_0 = N$ and $pr_0 < pr_k + k\,(1 \leq k \leq N)$, then with the choice $p = N/r_0$ we obtain the limiting values $S_0(a)t^{r_0}$ (t real). If r_0 is odd, this is a simply described straight line through the origin; if r_0 is even, it is a doubly described ray with endpoint at the origin. Otherwise, we choose $p > 0$ to satisfy $pr_0 = pr_j + j$ for some j and $pr_0 \leq pr_k + k\,(1 \leq k \leq N)$; in other words

$$p = \min \frac{j}{r_0 - r_j}, \tag{1.50}$$

where the minimum is taken over those $j\,(1 \leq j \leq N-1)$ for which $r_0 > r_j$. We obtain

$$R(x, a + tx^p) = \frac{F(x, t)}{x^{N-pr_0}}, \tag{1.51}$$

where $F(x, t)$ is polynomial in t of degree at most N, and contains fractional powers of x – in fact is polynomial in x and x^p. Note also that $F(0, t)$ is a polynomial in t of degree r_0, which contains only those powers t^{r_j} for which $pr_0 = pr_j + j$. Suppose now that $p = \mu/\nu$, where μ and ν are relatively prime natural numbers. Then with the substitution $x = s^\nu$ we obtain

$$R(s^\nu, a + ts^\mu) = \frac{A(s, t)}{s^{\nu N - \mu r_0}}, \tag{1.52}$$

where $A(s, t)$ is a polynomial in s and t and $A(0, t) = F(0, t)$ has degree r_0. Furthermore, since the powers t^{r_j} occurring in $A(0, t)$ satisfy $pr_0 = pr_j + j$, we obtain

$$r_0 - r_j = \frac{\nu j}{\mu}, \tag{1.53}$$

and, since μ and ν are relatively prime and $r_0 - r_j$ is an integer, μ divides j and so $r_0 - r_j$ is an integral multiple of ν. It follows that $A(0, t)$ has the form

$$A(0, t) = t^\alpha B(t^\nu), \tag{1.54}$$

where α is a non-negative integer and $B(t)$ is a polynomial in t of degree ≥ 1.

1.3.2

Now the form of R after this transformation is of the same general type in the variables s and t as it was in the variables x and y, namely a polynomial in s and t divided by a power of s. Therefore it is open to us to repeat this reducing process. Indeed, it follows from our comparison principle (see 1.3.6 below) that any finite limiting value of R will be attained along some curve $y = a + tx^p$, with $x \to 0$ and t remaining bounded, where p is a positive and indeed rational number. On the other hand $R \to \infty$ along such curves, if p is smaller than the above minimum choice. Therefore, finite limiting values can only be attained for either this or larger values of p. This implies that with our chosen minimum value of p and the above transformation, limiting values of R (associated with the zero a) will be attained with $s \to 0$ and t remaining bounded (in fact $t \to 0$ if the limiting value corresponds to a larger value of p). As before, each limiting value will correspond to a real zero, b say, of the polynomial $A(0, t)$. If the multiplicity of the zero b is s_0, then we make a transformation $t = b + us^q$, where q is chosen according to the same minimum process with which we chose p. Now clearly $s_0 \leq r_0 \leq N$; if $s_0 = r_0$, then $A(0, t) = c(t - b)^{r_0}$, where c is a non-zero constant, and therefore $A(0, t)$ contains all powers of t from 0 to r_0. It follows that $\nu = 1$ and so the power of s in the denominator is at most $N - r_0$. We see, therefore, that, if we continue repeating the process, at each step either the degree of the leading term strictly decreases or the power of the denominator strictly decreases by an integer amount. Thus eventually the process will terminate in the following way. Using the original notation $Q(x, y)/x^N$ as a standard form, the final transformation will take the form $y = a + tx^p$, where $p = N/r_0$. As earlier described either this will lead to a line or ray of asymptotic values or this choice of p will coincide with the minimum choice $p = \min(j/(r_0 - r_j))$. Following the transformation as before we obtain a curve of asymptotic values $A(0, t)$, a polynomial in the real variable t of degree $r_0 \geq 1$. Indeed, as the denominator is eliminated at the final step, the sequence of transformations will send our original polynomial P to a new polynomial $A(s, t)$. It is clear that for the existence of asymptotic values it is both necessary and sufficient that there exist such a finite sequence of transformations from P to another polynomial A. If we perform our construction for every real zero of our leading term at every stage, we will obtain all possible asymptotic values with $x > 0$ tending to ∞. We can obtain in this way a maximum of $N =$ degree of P asymptotic value curves, which are polynomial images of the real axis. Of course, if at one or other stage the leading term has no real zeros, then the polynomial tends to ∞ along the resulting tract. In a similar way, to obtain the asymptotic values of P with $x < 0$ tending to $-\infty$, we simply apply the same reasoning to the function $(-1)^N Q(-x, y)/x^N$, again

taking $x > 0$ and tending to 0. This gives another N possible asymptotic value curves.

1.3.3 Repetition of asymptotic values

However, as we shall see, each asymptotic value of P at ∞ is repeated along quite separate tracts, and in general each asymptotic value curve appears twice (except in one circumstance), either repeated or reversed. This is a consequence of the fact that, according to the above reasoning, the asymptotic values are attained as limits along rational algebraic curves of a particularly simple type. To be specific, if we put together the sequence of transformations from the initial $R(x, y)$ to the final polynomial $A(s, t)$, we obtain a single transformation of the form

$$x = s^q, \qquad y = \phi(s) + ts^\beta, \tag{1.55}$$

where q and β are natural numbers and where $\phi(s)$ is a real polynomial of degree $< \beta$; furthermore, the highest common factor of the powers of s appearing in the expression $\phi(s) + ts^\beta$ is 1. In short, we have

$$R(s^q, \phi(s) + ts^\beta) = A(s, t). \tag{1.56}$$

If $\phi(s) = \sum_{k=0}^{m} c_k s^k$, then the coefficients c_k are zeros of successive leading terms in our sequence of transformations. Note that the mapping $(s, t) \mapsto (x, y)$ is a 1–1 mapping of the half-plane $\{s > 0\}$ onto the half-plane $\{x > 0\}$. Indeed, this is clearly the case at each stage of the resolution process, which consists of blowing up a chosen zero into an entire line, the remainder of the line being pushed off to ∞. Furthermore, the mapping is 1–1 from $\{s < 0\}$ onto $\{x < 0\}$, if q is odd, and onto $\{x > 0\}$, if q is even. The curve $A(0, t)$ is a limiting curve of asymptotic values as $s \to 0$ through either positive values of s or negative values of s. If q is odd, the approach through negative values of s will correspond in the (x, y)-plane to a tract in $\{x < 0\}$ diametrically opposite the constructed tract in $\{x > 0\}$, and therefore the asymptotic value curve is repeated in a completely different portion of the plane. If q is even, the curve will again appear twice as limits along separate tracts in $\{x > 0\}$, unless $\phi(-s) = \phi(s)$.

This last possibility will occur if $\phi(s)$ is a polynomial in s^2 and β is odd. We will show that repetition of asymptotic values still occurs, though in a slightly strange way. Let d be the highest power of 2 which is a factor of q and the powers of s appearing in the expansion of ϕ. Then $x = s^{\sigma d}$, $y = \psi(s^d) + ts^\beta$, where $\psi(u)$ is a real polynomial in u and $\sigma d = q$. Now suppose that

$$\omega^d = 1 \quad \text{and} \quad \gamma = \overline{\omega}^\beta; \tag{1.57}$$

then

$$x = (\omega s)^{\sigma d}, \qquad y = \psi((\omega s)^d) + (\gamma t)(\omega s)^{\beta} \tag{1.58}$$

and so

$$A(\omega s, \gamma t) = A(s, t). \tag{1.59}$$

In particular, since d is even and β is odd, this holds when $\omega = -1$, $\gamma = -1$; i.e.

$$A(-s, -t) = A(s, t), \tag{1.60}$$

and in particular $A(0, -t) = A(0, t)$. It follows that our polynomial curve is an even function and so has the form $A(0, t) = B(t^2)$. The curve traced out is thus $B(t)$ from $+\infty$ to 0, followed by the reverse curve $B(t)$ from 0 to $+\infty$; i.e. we obtain half of $B(t)$ traversed twice. We show that in a separate tract the other half of $B(t)$ with $t < 0$ is similarly traversed. Firstly, note that either (i) $\sigma = q/d$ is odd, or (ii) σ is even. In case (i) we consider the tract given by

$$x = -s^{\sigma d} = (\lambda s)^{\sigma d}, \qquad y = \psi(-s^d) + ts^{\beta} = \psi((\lambda s)^d) + \eta t(\lambda s)^{\beta}, \tag{1.61}$$

where $\lambda^d = -1$ and $\eta = \overline{\lambda}^{\beta}$. Thus

$$R(x, y) = A(\lambda s, \eta t) \to A(0, \eta t) \text{ as } s \to 0. \tag{1.62}$$

Now $A(0, t) = A(0, \gamma t)$ and so the non-vanishing terms in the expansion of $A(0, t)$ involve only those powers t^k for which $\gamma^k = 1$; i.e. $\omega^{\beta k} = 1$. If $\omega = e^{2\pi i/d}$, this implies that d divides βk and so, as d is a power of 2 and β is odd, d divides k. Thus $A(0, t)$ has the form

$$A(0, t) = F(t^d), \tag{1.63}$$

where $F(t)$ is a real polynomial. Thus $A(0, \eta t) = F(\eta^d t^d) = F(-t^d)$. Thus with $s \to 0$ through positive values, $x \to 0$ through negative values; i.e. our tract lies in the negative half-plane and has limiting values the half-curve described by $F(t)$ with $t < 0$, as asserted. In case (ii) we consider the tract

$$x = s^{\sigma d} = (\lambda s)^{\sigma d}, \qquad y = \psi(-s^d) + ts^{\beta} = \psi((\lambda s)^d) + \eta t(\lambda s)^{\beta}, \tag{1.64}$$

and note that exactly the same reasoning applies, but our tract will now lie in the positive half-plane. However, in this case the polynomial $\psi(s)$ contains at least one odd power of s and so we obtain a distinct tract.

1.3.4

As we have seen, apart from the above case, the asymptotic value curve $A(0, t)$ appears twice corresponding to the distinct tract obtained by changing the

variable s to $-s$. The parameter β determines whether the curve is repeated or reversed as the tract is traversed in the positive direction: the positive direction corresponds to y increasing. If β is odd, $s \mapsto -s$ gives the tract

$$x = (-s)^q, \qquad y = \phi(-s) - ts^\beta \tag{1.65}$$

and for $s > 0$, y is clearly a decreasing function of t. Therefore, as the tract is traversed positively, we obtain the limiting curve $A(0, -t)$; i.e. the curve is reversed. On the other hand, when β is even, $s \mapsto -s$ gives $y = \phi(-s) + ts^\beta$ is an increasing function of t, and so the limiting curve is $A(0, t)$; i.e. the curve is repeated in the same direction.

1.3.5 Inverting the transformation

The mapping $x = s^q$, $y = \phi(s) + ts^\beta$ has the inverse

$$s = x^{1/q}, \qquad t = x^{-\beta/q}(y - \phi(x^{1/q})) \tag{1.66}$$

and so

$$R(x, y) = A(x^{1/q}, x^{-\beta/q}(y - \phi(x^{1/q}))). \tag{1.67}$$

This gives

$$P(x, y) = A(x^{-1/q}, x^{\beta/q}(y/x - \phi(x^{-1/q}))), \tag{1.68}$$

and so

$$P(x^q, y) = A(x^{-1}, -x^\beta \phi(x^{-1}) + yx^{\beta-q}) = A(x^{-1}, \psi(x) + yx^\alpha), \tag{1.69}$$

where $\alpha = \beta - q$ and, since ϕ has degree $\beta - h$, where $h \geq 1$, $\psi(x) = -x^\beta \phi(1/x)$ is a polynomial of the form $\psi(x) = cx^h + \cdots$, where $c \neq 0$. This may be regarded as the inverse of the formula

$$A(s, t) = P(s^{-q}, \phi(s)s^{-q} + ts^\alpha). \tag{1.70}$$

From these relations we see that on the curve $y = x^q \phi(x^{-1})$ we have

$$P(x^q, y) = A(x^{-1}, 0), \tag{1.71}$$

and, if $q \geq \beta - h$, y remains bounded on this curve as $x \to 0$. Thus $A(x^{-1}, 0)$ remains bounded as $x \to 0$; i.e. $A(x, 0)$ is a bounded polynomial as $x \to \infty$. This implies $A(x, 0)$ is constant, and so $P(x^q, y)$ is constant on the curve. Therefore, unless the polynomial $P(x, y)$ is constant on some curve,

$$q < \beta - h < \beta, \tag{1.72}$$

and in particular

$$\beta \geq q + 2 \quad \text{and} \quad \alpha > h. \tag{1.73}$$

To reiterate: knowing the tract equation $y = \phi(s)s^{-q} + ts^{\alpha}$, we obtain the polynomial $A(s, t)$ from $P(x, y)$; conversely, using the reverse tract equation $t = -s^{\beta}\phi(1/s) + ys^{\alpha}$, we can recover $P(x^q, y)$ from $A(s, t)$. However, if we work directly with $A(s, t)$, applying our process, but not assuming any prior knowledge of the asymptotic tracts, we will not necessarily obtain the above tract for t. The reason for this is that α need not be greater than the largest power of s in the expression $s^{\beta}\phi(1/s)$. No term s^{γ} with $\gamma \geq \alpha$ is needed in determining the asymptotic tract appropriate for $A(s, t)$ in the above direction; we will obtain a tract by deleting all terms involving those powers s^{γ} with $\gamma \geq \alpha$. On the other hand we can obtain precisely invertible transformations if ϕ has the form $\phi(x) = c_r x^r + \cdots$, where $c_r \neq 0$ and $r > q$; for then $\alpha > \deg(s^{\beta}\phi(1/s))$. Then the asymptotic tract is such that $y \to 0$ as $x \to \infty$. In this case, if the process is actually carried through, at each stage the value of p is always an integer and furthermore the zero is always unique and is a multiple zero of exact degree N; this is because the initial largest power of t is t^N, but also the final leading term is $P(0, y)$, which has exact degree N (by our initial assumption). Therefore the leading term has exact degree N at each stage. This phenomenon indicates that the possible polynomials $A(s, t)$ which can be obtained by applying the process to a polynomial P are of a rather special type. $A(s, t)$ has the asymptotic value curve $P(0, y)$ along a curve with $s \to \infty, t \to 0$; but all other asymptotic values of A, if any, will occur along curves where $s \to 0, t \to \infty$.

1.3.6 The comparison principle

We have omitted from our discussion of asymptotic values an important general principle, which establishes the algebraic nature of the asymptotic tracts and also the fact that, *for real analytic polynomials, the only finite limiting values at ∞ are asymptotic values*. The argument is easily sketched as follows.

Suppose we have a finite sum of terms, for which along a sequence the sum tends to a finite value, though some individual terms tend to ∞. We compare two terms by considering for a suitable global subsequence the ratio of the absolute values of the two terms. If the limit of the ratio is finite and non-zero, we say the terms are comparable. If the limit is zero, then the upper term of the ratio is smaller than the lower term; and *vice versa*. In this way we can order the groups of comparable terms and pick out the largest grouping in this ordering. It

is clear that the largest group must contain at least two terms, for if it contained only one, then this term would dominate the sum, and so the sum would tend to ∞. For polynomial terms the comparison of two such terms gives a tract of the type described above in which the sequence must lie.

1.3.7 Solving algebraic equations and algebraic functions

An algebraic function $f(x)$ is a function satisfying an equation of the form

$$P(x,\ f(x)) = 0, \tag{1.74}$$

where $P(x, y)$ is a real analytic polynomial. In other words $y = f(x)$ is a solution of the equation $P(x, y) = 0$. Of course, at a given x the equation may have no solutions or several solutions, though no more than the degree of P. We are interested in continuous solutions for an interval of values of x. We can use our procedure to construct an explicit form for a solution. After a normalisation we may assume that $P(0, 0) = 0$, and our aim is to construct a solution $f(x)$ with $f(0) = 0$. Let us attempt to find a solution for small $x > 0$; if such a solution exists then for an arbitrary natural number N the function $R_N(x, y) = P(x, y)/x^N$ has 0 as an asymptotic value as $x \to 0$; namely along the curve $y = f(x)$. Of course, our procedure is designed to find all limiting values of R_N as $x \to 0$; we obtain a finite number of algebraic curves $y = \phi_k(x) + tx^{\beta_k}$, which are polynomials in a fractional power of x, each of which reduces $R_N(x, y)$ to a polynomial in a fractional power of x and t; we then put $x = 0$ giving a polynomial in t, whose values for t real are the limiting values; we are concerned with the values of t which yield the limiting value 0, i.e. the zeros of the final polynomial, which is simply taking the procedure to the next stage. Thus 0 will be a limiting value of R_N along a finite number of curves $y = \psi_{k,N}(x)$. It is possible that for one, or more, of the $\psi_{k,N}$ we have $R_N(x, \psi_{k,N}(x)) = 0$; then $\psi_{k,N}(x)$ is a required solution and furthermore we have an explicit formula in powers of x for the solution. On the other hand any actual continuous solution $y = f(x)$ must satisfy for some k

$$f(x) = \phi_k(x) + t(x)x^{\beta_k}, \tag{1.75}$$

where $t(x)$ tends to a zero of the final polynomial as $x \to 0$; in other words one of the $\psi_{k,N}$ is an approximation to f. Since $\psi_{k,N}$ is a partial sum of $\psi_{k,N+r}$, we see that we obtain a series expansion for f by letting $N \to \infty$. At each stage the degree of the leading term does not increase, so eventually becomes a constant ν. Thus for large N the values of p will be integers and each leading term will

be a pure νth power of a linear polynomial giving just one multiple zero. This implies that the series expansion for f is a power series in $x^{1/q}$ for some natural number q. Our procedure continued indefinitely by increasing N will yield all the solutions $y = f(x)$ for $x > 0$ near 0 (i.e. all the *branches* of the algebraic function) and each such solution is a convergent power series, and therefore an analytic function, in a fractional power of x.