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Solitons, Nonlinear Evolution Equations and Inverse Scattering

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Chapter One

Introduction.

1.1 Historical Remarks and Applications.

"Solitons" were first observed by J. Scott Russell in 1834 [1838, 1844] whilst riding on horseback beside the narrow Union canal near Edinburgh, Scotland. There are a number of discussions in the literature describing Russell's observations. Nevertheless we feel that his point of view is so insightful and relevant that we present it here as well. He described his observations as follows:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulates round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called the Wave of Translation ... ."

Subsequently, Russell did extensive experiments in a laboratory scale wave tank in order to study this phenomenon more carefully. Included amongst Russell's results are the following:

1. he observed solitary waves, which are long, shallow, water waves of permanent form, hence he deduced that they *exist*; this is his most significant result;
2. the speed of propagation, c, of a solitary wave in a channel of uniform depth h is given by $c^2 = g(h + \eta)$, where η is the amplitude of the wave and g the force due to gravity.

Further investigations were undertaken by Airy [1845], Stokes [1847], Boussinesq [1871, 1872] and Rayleigh [1876] in an attempt to understand this phenomenon. Boussinesq and Rayleigh independently obtained approximate descriptions of the solitary wave; Boussinesq derived a one-dimensional nonlinear evolution equation, which now bears his name, in order to obtain his result.

These investigations provoked much lively discussion and controversy as to whether the inviscid equations of water waves would possess such solitary wave solutions. The issue was finally resolved by Korteweg and de Vries [1895]. They derived a nonlinear evolution equation governing long one dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\,\frac{\partial}{\partial \xi}\left(\tfrac{1}{2}\eta^2 + \tfrac{2}{3}\alpha\eta + \tfrac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right), \qquad \sigma = \tfrac{1}{3}h^3 - Th/(\rho g), \tag{1.1.1}$$

where η is the surface elevation of the wave above the equilibrium level h, α an small arbitrary constant related to the uniform motion of the liquid, g the gravitational constant, T the surface tension and ρ the density (the terms "long" and "small" are meant in comparison to the depth of the channel). The controversy was now resolved since equation (1.1.1), now known as the Korteweg-de Vries (KdV) equation, has permanent wave solutions, including ***solitary wave solutions*** (see §1.3 for details). Equation (1.1.1) may be brought into nondimensional form by making the transformation

$$t = \tfrac{1}{2}\sqrt{g/(h\sigma)}\,\tau, \qquad x = -\sigma^{-1/2}\xi, \qquad u = \tfrac{1}{2}\eta + \tfrac{1}{3}\alpha.$$

Hence, we obtain

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1.2}$$

where subscripts denote partial differentiations. Henceforth, we shall consider the KdV equation in this form (1.1.2) (note that any constant coefficient may be placed in front of any of the three terms by a suitable scaling of the independent and dependent variables). (1.1.2) may be thought of as the simplest "nonclassical" partial differential equation since it has the minimum number of independent variables, two; the lowest order not considered classically, that is three; the fewest terms of that order, one; the simplest such term, an unmixed derivative; the fewest number of terms containing the other derivative, which is of first order; the simplest structure for these terms, linear; and the simplest additional term to make the equation nonlinear, quadratic. (It might be thought that a simpler nonlinear term would be u^2, however the KdV equation has an extra symmetry with uu_x (Galilean invariance) and if u is interpreted as a velocity, then the convective derivative, which arises in continuum mechanics and is familiar to physicists and engineers, is $\mathrm{d}u/\mathrm{d}t = u_t + uu_x$.)

As Miura [1976] points out, despite this early derivation of the KdV equation, it was not until 1960 that any new application of the equation was discovered. Gardner and Morikawa [1960] rediscovered the KdV equation in the study of collision-free hydromagnetic waves. Subsequently the KdV equation has arisen in a number of other physical contexts, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, ... (for details and further references see, for example, the articles by Jeffrey and Kakutani [1972]; Scott, Chu and McLaughlin [1973]; Miura [1976] and monographs by Ablowitz and Segur [1981]; Calogero and Degasperis [1982]; Dodd, Eilbeck, Gibbon and Morris [1982]; Lamb [1980]; Novikov, Manakov, Pitaevskii and Zakharov [1984] — for a discussion concentrating primarily on the pre-1965 history of the KdV equation, see Miles [1980b]).

It has been known for a long time that the KdV equation (1.1.2) possesses the solitary wave solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2 \left\{ \kappa(x - 4\kappa^2 t - x_0) \right\}, \tag{1.1.3}$$

where κ and x_0 are constants (in fact this solution was known to Korteweg and de Vries). Note that the velocity of this wave, $4\kappa^2$, is proportional to the amplitude, $2\kappa^2$; therefore taller waves travel faster than shorter ones. Zabusky and Kruskal [1965] discovered that these solitary wave solutions have the remarkable property that the interaction of two solitary wave solutions is elastic, and are called *solitons* (see §1.4 details).

The solitons observed by Russell were small amplitude surface waves. There have been several investigations examining the validity of the KdV equation (1.1.2) as a model of the evolution of small amplitude water waves as they propagate in one direction in shallow water. These studies have compared the solutions of (1.1.2) with experimental results (see, for example, Hammack and Segur [1974, 1978]). In physical terms the KdV equation arises if the water waves are strictly one-dimensional, that is one spatial dimension and time (for a derivation of the KdV equation see, for example, Chapter 4 of Ablowitz and Segur [1981]).

In many physical situations, internal waves can arise at the interface of two layers of fluid due the gravitational effects in a stably stratified fluid. Several theoretical models exist which govern the evolution of long internal waves with small amplitudes in a stably stratified fluid including the KdV equation (1.1.2), the intermediate long wave (ILW) equation

$$u_t + \delta^{-1} u_x + 2uu_x + \mathrm{T}u_{xx} = 0, \tag{1.1.4a}$$

where $\mathrm{T}u$ is the singular integral operator

$$(\mathrm{T}f)(x) = \frac{1}{2\delta} \fint_{-\infty}^{\infty} \coth \left\{ \frac{\pi}{2\delta}(y - x) \right\} f(y)\, \mathrm{d}y, \tag{1.1.4b}$$

with $\fint_{-\infty}^{\infty}$ the Cauchy principal value integral (Joseph [1977]; Kubota, Ko and Dobbs [1978]), and the Benjamin-Ono (BO) equation

$$u_t + 2uu_x + \mathrm{H}u_{xx} = 0, \tag{1.1.5a}$$

where $\mathrm{H}u$ is the Hilbert transform

$$(\mathrm{H}f)(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(y)}{y - x}\, \mathrm{d}y \tag{1.1.5b}$$

(Benjamin [1967]; Davies and Acrivos [1967]; Ono [1975]). In the shallow water limit, as $\delta \to 0$, (1.1.4) reduces to the KdV equation

$$u_t + 2uu_x + \tfrac{1}{3}\delta u_{xxx} = 0, \tag{1.1.6}$$

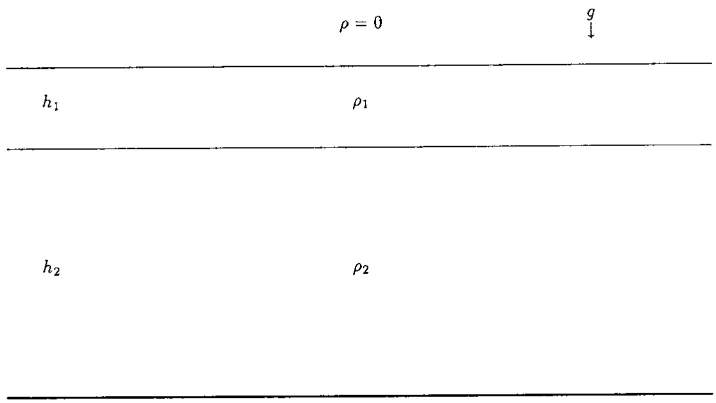

FIGURE 1.1.1 *The two-layer configuration.*

and in the deep-water limit as $\delta \to \infty$, to the BO equation. Therefore the ILW equation (1.1.4) may be thought of as being an equation intermediate between (1.1.5) and (1.1.6).

Consider two incompressible, immiscible fluids, with densities and depths h_1, h_2 ($h := h_1 + h_2$) with the lighter fluid, of height h_1, lying over a heavier fluid of height h_2, in a constant gravitational field (Figure 1.1.1). The lower fluid rests on a horizontal impermeable bed, and the upper fluid is bounded by a free surface.

Suppose the the characteristic wave amplitude is denoted by a and the characteristic wavelength by $\lambda = k^{-1}$. The basic assumptions for the derivation of the KdV equation (1.1.6), the ILW equation (1.1.4) and the BO equation (1.1.5), as models for internal waves are (see Chapter 4 of Ablowitz and Segur [1981] for details).

Korteweg de-Vries equation:

(A1) the waves are long waves in comparison with the total depth, $kh \ll 1$;
(A2) the amplitude of the waves is small, $\varepsilon = a/h \ll 1$;
(A3) the two effects (A1) and (A2) approximately balance, i.e., $kh = O(\varepsilon)$;
(A4) viscous effects may be neglected.

Intermediate-Long-Wave equation:

(B1) there is a thin (upper) layer, $\varepsilon = h_1/h_2 \ll 1$;
(B2) the amplitude of the waves is small, $a \ll h_1$;
(B3) the two effects (B1) and (B2) balance, i.e., $a/h_1 = O(\varepsilon)$;
(B4) the characteristic wavelength is comparable to the total depth of the fluid, $kh = O(1)$;
(B5) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$ [this is implied by (B1) and (B4)];
(B6) viscous effects may be neglected.

Note that in the ILW equation (1.1.4), the parameter δ is effectively kh. For derivations of the ILW equation (1.1.4) in which the fluid is confined between two rigid walls see Kubota, Ko and Dobbs [1978]; Segur and Hammack [1982] (these two derivations are slightly different).

Benjamin-Ono equation:

(C1) there is a thin (upper) layer, $h_1 \ll h_2$;

(C2) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$;

(C3) the waves are short in comparison with the total depth of the fluid, $kh \gg 1$;

(C4) the amplitude of the waves is small, $a \ll h_1$;

(C5) viscous effects may be neglected.

Under these assumptions one obtains the KdV, ILW and BO equations, after a suitable scaling. Segur and Hammack [1982] have examined the validity of the KdV and ILW equations as models for internal waves by comparing the theoretical solutions of the KdV and ILW equations with experimental results.

Recently there has been considerable interest in the observation of what scientists think might be solitons in the oceans. The availability of photographs taken from satellites and spacecraft orbiting the earth have greatly assisted in these observations. Peculiar striations, visible on satellite photographs of the surface of the Andaman and Sulu seas in the Far East, have been interpreted as secondary phenomena accompanying the passage of "internal solitons", solitary wavelike distortions of the boundary layer between the warm upper layer of sea water and cold lower depths. These internal solitons are travelling ridges of warm water, extending hundreds of meters down below the thermal boundary, and enormous energies which they carry are presumed to be the cause of unusually strong underwater currents experienced by deep-sea drilling rigs. In order to continue deep-sea drilling for oil in areas where these internal solitons occur, the drilling rigs will have to be built to withstand these forces.

Osborne and Burch [1980] (see also Osborne [1990]) investigated the underwater currents which were experienced by an oil rig in the Andaman sea, which was drilling at a depth of 3600ft (one drilling rig was apparently spun through ninety degrees and moved one hundred feet by the passage of a soliton below). Satellite photographs had shown that there were 100km long striations on the Andaman sea, separated by 6 to 15km and grouped in packets of typically 4 to 8. The average time between the arrival of these packets at the research vessel was 12 hours 26 minutes, suggesting that this was some kind of tidal phenomenon. Osborne and Burch spent four days measuring underwater currents and temperatures. The striations seen on satellite photographs turned out to be kilometer-wide bands of extremely choppy water, stretching from horizon to horizon, followed by about two kilometers of water "as smooth as a millpond". These bands of agitated water are called "tide rips", they arose in packets of 4 to 8, spaced about 5 to 10km apart (they reached the research vessel at approximately hourly intervals) and this pattern was repeated with the regularity of tidal phenomenon.

Osborne and Burch [1980] found that the amplitude of each succeeding soliton was less than the previous one, which, as we shall see in §1.7 below, is precisely what is expected for solitons (recall that the velocity of a solitary wave solution (1.1.3) of the KdV equation increases with amplitude). Thus if a number of solitons are generated together, then we expect them eventually to be arranged in an ordered sequence of decreasing amplitude. From the spacing between successive waves in a packet and the rate of separation calculated from the KdV equation, Osborne and Burch were able to estimate the distance the packet had travelled from its source and thus identify possible source regions. They concluded that the solitons are generated by tidal currents off northern Sumatra or between the islands of the Nicobar chain that extends beyond it.

Underwater measurements showed a rapid circulation of water associated with the solitary waves. The interaction of this internal circulation with ordinary surface waves produces the tide rip and appear to explain the origin of the forces experienced by the oil drilling rigs. Osborne and Burch [1980] conclude that, despite the irregular geometry of the Andaman sea, their observations have good general agreement with the predictions for internal solitons as given by the KdV equation.

Apel and Holbrook [1980] (see also Holbrook, Apel and Tsai [1980]) undertook a detailed study of internal waves in the Sulu sea. Satellite photographs had suggested that the source of these waves was near the southern end of the Sulu sea and their research ship followed one wave packet for more than 250 miles over a period of two days — an extraordinary coherent phenomenon. These internal solitons travel at speeds of about 8 kilometers per hour (5 miles per hour), with amplitude of about 100 meters and wavelength of about 1700 meters.

Further observations and studies of solitons in oceans include the Strait of Messina (Alpers and Salustri [1983]; Santoleri [1983]); the Strait of Gibraltar (Lacombe and Richez [1982]); off the western side of Baja California (Apel and Gonzalez [1983]); the Gulf of California (Fu and Holt [1984]); the Archipelago of La Maddalena (Manzella, Bohm and Salustri [1983]) and the Georgia Strait (Hughes and Gower [1983]).

Rossby waves (Rossby [1939]) are long waves between layers of the atmosphere, created by the rotation of the planet. As one might suspect, there is an analogy between internal waves and Rossby waves under suitable conditions. The KdV equation has been proposed as a model for the evolution of Rossby waves (Long [1964]; Benney [1966]; Redekopp [1977]; Redekopp and Weidman [1978] — see also Miles [1980b] and the references cited therein). There has been been a conjecture (Maxworthy and Redekopp [1976]) that Jupiter's Great Red Spot might be a solitary Rossby wave, though recent work seems to indicate that the red spot is a result of strong differential rotation (cf. Meyers, Sommeria and Swinney [1989]).

We have noted that the KdV equation (1.1.2) can describe the evolution of small amplitude water waves as they propagate in shallow water; the evolution described being weakly nonlinear and weakly dispersive, where these effects are of the same order. These solitary waves are usually regarded as propagating in a uniform rectangular channel. There have been several studies on the problem of quasi one-dimensional solitary waves in channels of varying width, depth or shape of the channel (for a review see Miles [1980b] and the references cited therein).

A further restriction in the application of the KdV equation as a practical model for water waves, is that the KdV equation is strictly only one-dimensional (that is one spatial-dimension plus time), whereas the surface is two-dimensional. A two-dimensional generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{1.1.7}$$

where $\sigma^2 = \pm 1$ (Kadomtsev and Petviashvili [1970]). Whereas the KdV equation (1.1.2) describes the evolution of long water waves of small amplitude if they are strictly one-dimensional, the KP equation (1.1.7) describes their evolution if they are weakly two-dimensional (in §1.2 below, we give a derivation of the KP equation as a model for surface waves). The evolution described by the KP equation is weakly nonlinear, weakly dispersive and weakly two-dimensional with all three effects being of the same order; the choice of sign depends on the relevant magnitude of gravity and surface tension (cf. §1.2). The KP equation has also been proposed as a model for surface waves and internal waves in straits or channels of varying depth and width (see, for example, Santini [1981]; David, Levi and Winternitz [1987a, 1989]).

The KdV equation (1.1.2), the ILW equation (1.1.4), the BO equation (1.1.5) and the KP equation (1.1.7) have been extensively studied in recent years, primarily by mathematicians and physicists. We have already seen that they arise in the description of physically interesting phenomena, however much of the interest in these nonlinear evolution equations is due to the fact that they are thought of as being *completely integrable*, or *exactly solvable* equations. This terminology is a consequence of the fact that the initial value problem for each of these nonlinear equations can be solved exactly by a method employing *inverse scattering*, which we shall study in depth in these notes. The inverse scattering method was discovered by Gardner, Greene, Kruskal and Miura [1967] as a method of solving the initial value problem for the KdV equation (on the infinite line), for initial values that decay sufficiently rapidly at infinity. Subsequently numerous other physically interesting equations in one spatial dimension were solved by generalizations of this technique, which is now referred to as the *Inverse Scattering Transform* (I.S.T.), for example the nonlinear Schrödinger, Sine-Gordon, three-wave interaction, Modified KdV and Boussinesq equations. The I.S.T. scheme for the ILW (1.1.4) and the BO equation (1.1.5), which are integro-differential equations, was derived by Kodama, Ablowitz and Satsuma [1982] and

Fokas and Ablowitz [1983b], respectively. For the KP equation (1.1.7), the I.S.T. scheme is dependent upon the choice of sign for σ^2. The scheme for the KP equation with $\sigma^2 = -1$ (which is known as KPI) was derived by Manakov [1981] and Fokas and Ablowitz [1983d]; and the scheme in the case $\sigma^2 = 1$ (which is known as KPII), by Ablowitz, BarYaacov and Fokas [1983]. (As we show in Chapter 5, the I.S.T. schemes for KPI and KPII are fundamentally different.)

The organization of these notes is as follows. In Chapter 1 we give an introduction to several of the most important and significant aspects in the development of the I.S.T. schemes for nonlinear evolution equations. In Chapters 2 and 3 we discuss I.S.T. schemes for one-dimensional equations; discussing the KdV equation (1.1.2) in Chapter 2 and more general I.S.T. schemes in one dimension in Chapter 3 (including the I.S.T. schemes for differential-difference and partial-difference equations). In Chapter 4 we discuss the I.S.T. scheme for integro-differential equations using the ILW equation (1.1.4) and the BO equation (1.1.5) as the prototype examples. In Chapters 5 and 6 we discuss multidimensional I.S.T. schemes; in Chapter 5 discussing inverse scattering in two spatial dimensions respectively, using the KP equation (1.1.7) as prototype example and in Chapter 6 discussing inverse scattering in higher dimensions. In Chapter 7 we discuss several topics concerning integrability associated with the Painlevé equations, which are six second order, nonlinear ordinary differential equations; in particular, the so-called Painlevé tests and inverse scattering schemes for the Painlevé equations using the second Painlevé equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y^3 + xy + \alpha, \tag{1.1.8}$$

where α is a constant, as the prototype example. Finally, in Chapter 8, we make some concluding remarks and mention some difficult open problems.

1.2 Physical Derivation of the KP Equation.

In this section we present a physical derivation of the KP equation (1.1.7) as a model for surface waves.

The classical problem of water waves is to find the irrotational motion of an inviscid, incompressible, homogeneous fluid, subject to a constant gravitational force g. The fluid rests on a horizontal and impermeable bed of infinite extent at $z = -h$ and has a free surface at $z = \eta(x, y, t)$. Since the fluid is irrotational and incompressible, then it has a velocity potential ϕ satisfying

$$\nabla^2 \phi = 0, \qquad -h < z < \eta(x, y, t). \tag{1.2.1a}$$

It is subject to the boundary conditions on the bottom $z = -h$

$$\phi_z = 0 \tag{1.2.1b}$$

(since the bed is impermeable), and on the free surface $z = \eta$

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} \equiv \eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \tag{1.2.1c}$$

(kinematic condition) and

$$\phi_t + g\eta + \tfrac{1}{2}|\nabla\phi|^2 = \frac{T}{\rho}\,\frac{\eta_{xx}(1+\eta_y^2) + \eta_{yy}(1+\eta_x^2) - 2\eta_{xy}\eta_x\eta_y}{(1+\eta_x^2+\eta_y^2)^{3/2}} \tag{1.2.1d}$$

(dynamic condition), where T is the surface tension and ρ is the density of the fluid. Boundary conditions in (x, y) and initial conditions are also required. For isolated waves, then $\nabla\phi$ and η should vanish as $(x^2 + y^2) \to \infty$. (In other problems, periodic boundary conditions in x and y may be appropriate.)

This problem, first posed by Stokes [1847], is nonlocal, highly nonlinear and not surprisingly remains unsolved in its general form. To make further progress, we need to impose additional assumptions on the solutions to equations (1.2.1). The first such assumption is that the wave amplitudes should be small. If we interpret small to mean infinitesimal, then we may linearize equation (1.2.1) about $\nabla\phi = 0$, $\eta = 0$, and seek solutions of the linearized equations proportional to $\exp\{\mathrm{i}(kx + my - \omega t)\}$ (see Lamb [1932,§§228,266,267]). The result is the linearized dispersion relation

$$\omega^2 = \left(g\kappa + \kappa^3 T/\rho\right)\tanh(\kappa h), \tag{1.2.2}$$

where $\kappa^2 := k^2 + m^2$. From this one computes the group velocity and shows that the linearized problem is dispersive at most wave numbers, but not as $\kappa \to 0$ (i.e., long waves, or shallow water waves), where it is only weakly dispersive. The KdV and KP equations arise as models of the water wave problem in this weakly dispersive limit $\kappa h \ll 1$.

We orient the horizontal coordinates so that the x-direction is the principal direction of wave propagation. To derive the KP equation (1.1.7) we assume that:

1. wave amplitudes are small, $\varepsilon = |\eta|_{\max}/h \ll 1$;
2. the water is shallow relative to typical horizontal wavelengths, $(\kappa h)^2 \ll 1$;
3. the waves are nearly one-dimensional, $(m/k)^2 \ll 1$;
4. these three effects all have comparable influence (i.e., they balance), $(m/k)^2 = O((\kappa h)^2) = O(\varepsilon)$.

These assumptions imply a certain scaling of the original equation (see below); an evolution equation which follows may then be found using a multiple scales method.

At leading order, the equations are linear (from 1), nondispersive (from 2) and one-dimensional (from 3), the result being the linear wave equation

$$\eta_{tt} - gh\eta_{xx} = 0. \tag{1.2.3}$$

At this order every wave has permanent form, not because it is soliton but because we are solving the one-dimensional, linear wave equation.

When the perturbation expansion is continued to second order, the one-dimensional, linear wave equation has homogeneous (forcing) terms representing weak nonlinearity, weak dispersion and weak two-dimensionality. Each of these effects contributes to a secular term at second order. Define

$$r = \varepsilon^{1/2}\,\frac{x-\sqrt{gh}\,t}{h}, \quad s = \varepsilon^{1/2}\,\frac{x+\sqrt{gh}\,t}{h}, \quad \zeta = \varepsilon\,\frac{y}{h}, \quad \tau = \varepsilon^{3/2}\,\frac{\sqrt{gh}\,t}{6h}, \tag{1.2.4a}$$

and seek solutions of the form

$$\eta(x,y,t;\varepsilon) = \tfrac{2}{3}\varepsilon h\,[u(r,\zeta,\tau) + v(s,\zeta,\tau)] + O(\varepsilon^2 h). \tag{1.2.4b}$$

Since we are interested in problems where the initial disturbances are localized, then it is convenient to assume *a fortiori* that the physical quantities have compact support initially. In this case it is easily shown that u and v in (1.2.4b) have compact support as well. No secular terms arise at second order provided that $U(r,\zeta,\tau)$ and $U(s,\zeta,\tau)$ satisfy

$$[U_\tau + 6UU_r + (1-\hat{T})U_{rrr}]_r + 3U_{\zeta\zeta} = 0, \tag{1.2.5a}$$

$$[V_\tau - 6VV_s - (1-\hat{T})V_{sss}]_s - 3V_{\zeta\zeta} = 0, \tag{1.2.5b}$$

where $\hat{T} := T/(3\rho g h^2)$ is the dimensionless surface tension. Thus the left and right running waves each evolve according to their own KP equation, which describe how the two sets of waves each interact with themselves over a long time scale

$$\tau = \tfrac{1}{6}\varepsilon^{3/2}\,\sqrt{g/h}\,t = O(1).$$

To make the model quite explicit, we also write the KP equation for the right-going waves in its dimensional form for $\eta(x,y,t)$:

$$\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{gh}}\,\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \left(\frac{3h^2\rho g - T}{18\rho g}\right)\eta_{xxx}\right] + \tfrac{1}{2}\eta_{yy} \sim 0. \tag{1.2.6}$$

For most cases of interest in water waves $1-\hat{T} > 0$ (in fact usually $\hat{T} \ll 1$ and so $\hat{T}$ may be neglected), which corresponds to $\sigma^2 = 1$ in equation (1.1.7) (i.e., KPII). Thus from equation (1.2.2), the linearized phase speed is a (local) maximum at $\kappa = 0$. In this case, equation (1.2.6) is equivalent to equation (1.1.7) with $\sigma = +1$, i.e.

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1.2.7}$$

which is usually called KPII.

Obviously every solution of the KdV equation (1.1.2) is also a solution of the KPII equation (1.2.7). More generally, Satsuma [1976] showed that the KPII equation (1.2.7) has N line-soliton solutions, with the N line-solitons travelling in N different directions and interacting obliquely. In particular, a one line-soliton solution is

$$u(x,y,t) = 2\kappa^2 \operatorname{sech}^2 \left\{\kappa[x + \lambda y - (4\kappa^2 + 3\lambda^2)t + \delta_0]\right\}. \tag{1.2.8}$$

Essentially these are one-dimensional solutions travelling at an angle to the y-axis. A two line-soliton (written in Hirota's notation — see §2.6.5 below) is

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln F(x,y,t), \tag{1.2.9a}$$

with

$$F(x,y,t) = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}), \tag{1.2.9b}$$

where

$$\eta_i = 2\kappa_i[x + \lambda_i y - (4\kappa_i^2 + 3\lambda_i^2)t] + \delta_i, \tag{1.2.9c}$$

$$\exp(A_{12}) = \frac{4(\kappa_1 - \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}{4(\kappa_1 + \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}. \tag{1.2.9d}$$

One and two line-soliton solutions of the KPII equation are illustrated in Figures 1.2.1 and 1.2.2, respectively. Note that in Figure 1.2.2, away from the interaction region, each wave is essentially a KdV-type soliton (as expected, there is a phase shift due to the interaction). We remark that solutions such as (1.2.8) and (1.2.9) do not decay as $(x^2 + y^2)^{1/2} \to \infty$ in all directions.

For very thin sheets of water $\hat{T} > 1$ (when surface tension dominates gravity), then equation (1.2.6) is equivalent to equation (1.1.7) with $\sigma^2 = -1$, i.e.

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.2.10}$$

which is usually called KPI. Here one-dimensional solitons are unstable (Kadomtsev and Petviashvili [1970]). (We remark that the KP equation with $\hat{T} > 1$ also applies if we neglect gravity, remove the horizontal bed (and the viscous boundary layer it creates) on which the thin sheet of water lies and restrict out attention to the symmetric modes so that $\phi_z = 0$ holds.) In this case, there exist "lump" solutions of the KPI equation,

$$u(x,y,t) = 4\,\frac{\left\{-\left[x + \lambda y + 3(\lambda^2 - \mu^2)t\right]^2 + \mu^2(y + 6\lambda t)^2 + 1/\mu^2\right\}}{\left\{\left[x + \lambda y + 3(\lambda^2 - \mu^2)t\right]^2 + \mu^2(y + 6\lambda t)^2 + 1/\mu^2\right\}^2}, \tag{1.2.11}$$

which decay algebraically as $(x^2 + y^2)^{1/2} \to \infty$ (cf. Manakov, Zakharov, Bordag, Its and Matveev [1977]; Satsuma and Ablowitz [1979]).

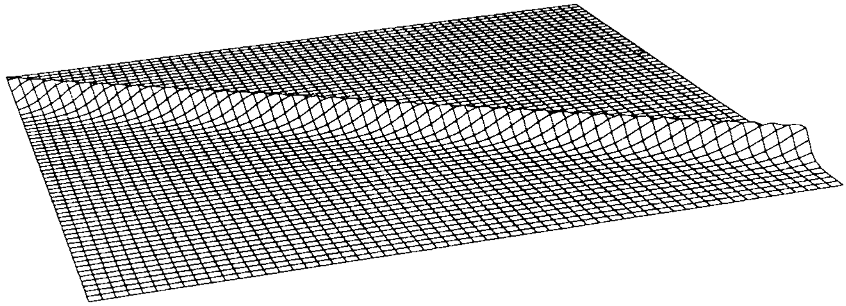

FIGURE 1.2.1 *One line-soliton solution of the KPII equation* (1.2.7).

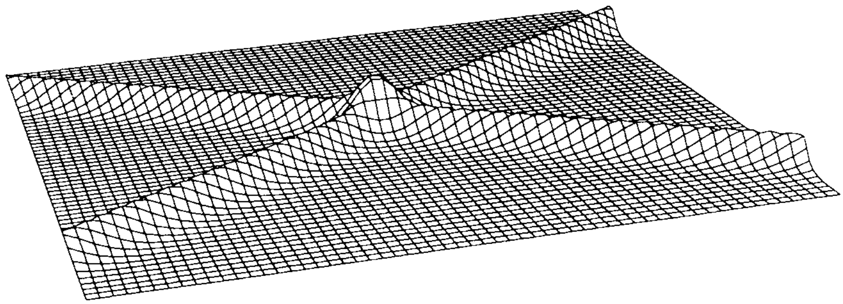

FIGURE 1.2.2 *Two line-soliton solution of the KPII equation* (1.2.7).

1.3 Travelling Wave Solutions of the KdV Equation.

The first interesting property of the KdV equation is the existence of permanent wave solutions, including solitary wave solutions.

DEFINITION 1.3.1

A solitary wave solution of a partial differential equation

$$\Delta(x,t,u) = 0,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and $u \in \mathbb{R}$ the dependent variable, is a travelling wave solution of the form

$$u(x,t) = w(x - \gamma t) = w(z), \tag{1.3.1}$$

whose transition is from one constant asymptotic state as $z \to -\infty$ to (possibly) another constant asymptotic state as $z \to \infty$. (Note that some definitions of solitary waves require the constant asymptotic states to be equal — often to zero.)

To obtain travelling wave solutions of the KdV equation, we seek a solution in the form (1.3.1) which yields a third order ordinary differential equation for w

$$w''' + 6ww' - \gamma w' = 0, \tag{1.3.2}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$. Integrating this twice gives

$$\tfrac{1}{2}(w')^2 = f(w) := -\tfrac{1}{2}(2w^3 - \gamma w^2 - Aw - B), \tag{1.3.3}$$

with A, B constants. Since we are interested in obtaining real, bounded solutions for the KdV equation, then we require that $f(w) \geq 0$ and so we study the zeros of $f(w)$. There are two cases to consider: (1) when $f(w)$ has only one real zero and (2) when $f(w)$ has three real zeros (for which there are three subcases).

Case 1. If $f(w)$ has only one real zero, a, then it is one of the two forms shown in Figure 1.3.1. If $w'(0) < 0$, then $f(w) > 0$ for all $z > 0$, and so w decreases monotonically to $-\infty$ as $z \to \infty$. If $w'(0) > 0$, then w increases until it reaches a, at say z_1 (that is $w(z_1) = a$), which is a simple maximum of w. Therefore $w'(z) < 0$, for $z > z_1$ and thereafter w decreases monotonically to $-\infty$ as $z \to \infty$. Hence there are no bounded solutions in this case.

Case 2. If $f(w)$ has three real zeros, a, b and c, then we may assume that $a \leq b \leq c$, and so we may write

$$f(w) = -(w - a)(w - b)(w - c), \tag{1.3.4}$$