1 Fundamentals

1.1 Notation and table of symbols

Except where noted, the following symbols will be used consistently throughout this work:

- $a = \beta / \alpha$: Ratio of S- and P-wave velocities
- $b = b_j \hat{e}_j$: Body load vector
- $C_R$: Rayleigh-wave velocity
- $C_n$: $3 \times 3$ Bessel matrix, cylindrical coordinates and flat layers (see Table 10.2)
- $g_{ij}$: Green’s function vector for the frequency domain response due to a unit load in direction $j$
- $F_n$: $3 \times 3$ traction matrix, cylindrical coordinates (see Table 10.4)
- $F_n^{(1)}, F_n^{(2)}$: As $F_n$ above, assembled with first and second Hankel functions
- $\mathcal{F}$: Fourier transform operator
- $\mathcal{F}^{-1}$: Inverse Fourier transform operator
- $g_{ij}$: Green’s function for the frequency domain response in direction $i$ due to a unit load in direction $j$
- $g_{ij,k} = \partial g_{ij}/\partial x_k$: Derivative with respect to the receiver location
- $g_{ij,k} = \partial g_{ij}/\partial x_k'$: Derivative with respect to the source location
- $H_n^{(1)}(kr), H_n^{(2)}(kr)$: First and second spherical Hankel functions of order $n$
- $H_n^{(1)}(kr), H_n^{(2)}(kr)$: First and second Hankel functions of order $n$ (Bessel functions of the third kind)
- $H_n$: $3 \times 3$ displacement matrix, cylindrical coordinates (see Table 10.3)
- $H_n^{(1)}, H_n^{(2)}$: As $H_n$ above, assembled with first and second Hankel functions
- $H_n$: $3 \times 3$ spherical Bessel matrix, spherical coordinates (see Table 10.7)
\[ \mathcal{H}(t - t_u) = \begin{cases} 0 & t < t_u \\ \frac{1}{2} & t = t_u \\ 1 & t \geq t_u \end{cases} \]

Unit step function, or Heaviside function

\[ i = \sqrt{-1} \]

Imaginary unit (non-italicized)

\[ i, j, k \]

Sub-indices for the numbers 1, 2, 3 or coordinates \( x, y, z \)

\[ \hat{e}_1, \hat{e}_2, \hat{e}_3 \]

Orthogonal unit basis vectors in Cartesian coordinates

\[ J_n(kr), Y_n(kr) \]

Bessel functions of the first and second kind

\[ J_3 \times 3 \text{ spherical orthogonality condition (Section 9.3, Table 10.6)} \]

\[ k \]

Radial wavenumber

\[ k_P = \frac{\omega}{\alpha} \]

P wavenumber

\[ k_S = \frac{\omega}{\beta} \]

S wavenumber

\[ k_z \]

Vertical wavenumber

\[ k_P = \sqrt{k^2 - k_P^2} \]

Vertical wavenumber for P waves, flat layers

\[ k_S = \sqrt{k^2 - k_S^2} \]

Vertical wavenumber for S waves, flat layers

\[ k_m = \sqrt{k_m^2 - k_z^2} \]

Radial wavenumber for P waves, cylindrical layers

\[ k_m = \sqrt{k_m^2 - k_z^2} \]

Radial wavenumber for S waves, cylindrical layers

\[ L_m^3 \times 3 \text{ Spheroidal (co-latitude) matrix (see Tables 10.7, 10.8)} \]

\[ M \]

Intensity of moment, torque, or seismic moment

\[ M_{ij} \]

Displacement in direction \( i \) due to seismic moment with axis \( j \).

\[ p \]

Load vector

\[ \tilde{p} \]

Load vector in frequency–wavenumber domain

\[ p \]

As above, but vertical component multiplied by \(-i = -\sqrt{-1}\)

\[ p = \sqrt{1 - (k_P/k)^2} \]

Dimensionless vertical wavenumber for P waves

\[ P \]

Load amplitude

\[ P_m \]

Legendre function (polynomial) of the first kind

\[ P_n^m \]

Associated Legendre function of the first kind

\[ Q_n^m \]

Associated Legendre function of the second kind

\[ R \]

Source–receiver distance in 3-D space

\[ r \]

Source–receiver distance in 2-D space, or range

\[ r, \theta, z \]

Cylindrical coordinates (see Fig. 1.2)

\[ \hat{f}, \hat{i}, \hat{k} \]

Orthogonal unit basis vectors in cylindrical coordinates

\[ \hat{r}, \hat{\theta}, \hat{z} \]

Orthogonal unit basis vectors in spherical coordinates

\[ R, \phi, \theta \]

Spherical coordinates (see Fig. 1.3)

\[ s = \sqrt{1 - (k_S/k)^2} \]

Dimensionless vertical wavenumber for S waves

\[ t \]

Time

\[ t_P = r/\alpha \]

P-wave arrival time

\[ t_S = r/\beta \]

S-wave arrival time
1.1 Notation and table of symbols

$T_R = r/C_R$ Rayleigh-wave arrival time

$T$ Torque

$T_n$ Azimuthal matrix

$T_{ij}$ Displacement in direction $i$ due to a unit torque with axis $j$.

$\mathbf{u}$ Displacement vector

$\mathbf{\hat{u}}$ Displacement vector in frequency-wavenumber domain

$\mathbf{\hat{u}}$ As above, but vertical component multiplied by $-i = -\sqrt{-1}$

$u_x, u_y, u_z \equiv u_1, u_2, u_3$ Displacement components in Cartesian coordinates

$u_r, u_\theta, u_z \equiv u_4, u_5, u_6$ Displacement components in cylindrical coordinates

$u_{sz}$ or $u_{tij}$, etc. Displacement in direction $x$ (or $i$) due to force in direction $z$ (or $j$)

$U_{ij}, u_{ij}$ Green’s function for the time domain response in direction $i$ due to a unit load in direction $j$

$x, y, z \equiv x_1, x_2, x_3$ Cartesian coordinates (see Fig. 1.1)

$x', y', z'$ Cartesian coordinates of the source

$\alpha = \beta \sqrt{2(1 - \nu)/(1 - 2\nu)}$ P-wave velocity

$\beta = \sqrt{\mu/\rho}$ S-wave velocity

$\gamma_i$ Direction cosine of $R$ with $i$th axis (see Cartesian coordinates)

$\delta(t - t_S) = \frac{d\delta(t - t_S)}{dt} = \begin{cases} 0, & t < t_S \\ \infty, & t = t_S \\ 0, & t > t_S \end{cases}$ Dirac-delta singularity function

$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ Kronecker delta

$\lambda$ Lamé constant

$\lambda + 2\mu = \rho \alpha^2$ Constrained modulus

$\mu = \rho \beta^2$ Shear modulus

$\nu$ Poisson’s ratio

$\tau = t\beta/r = t/t_S$ Dimensionless time

$\rho$ Mass density

$\theta$ Azimuth in cylindrical and spherical coordinates

$\theta_i$ Angle between $R$ and the $ith$ axis ($\gamma_i = \cos \theta_i$), $i = 1, 2, 3$

$\Phi$ Dilatational Helmholtz potential

$\chi = \chi(\omega)$ Dimensionless component function of Green’s functions (in later chapters, a Helmholtz shear potential)

$X = X(t)$ Inverse Fourier transform of $\chi(\omega)$ (or the convolution of the latter with an arbitrary time function)

$\psi = \psi(\omega)$ Dimensionless component function of Green’s functions (in later chapters, a Helmholtz shear potential)

$\Psi = \Psi(t)$ Inverse Fourier transform of $\psi(\omega)$ (or the convolution of the latter with an arbitrary time function)
1.2 Sign convention

Component of vectors, such as displacements and forces, are always defined positive in the positive coordinate directions, and plots of displacements are always shown upright (i.e., never reversed or upside down).

Point and line sources will usually – but not always – be located at the origin of coordinates. When this is not the case, it will be indicated explicitly.

Wave propagation in the frequency–wavenumber domain will assume a dependence of the form $\exp(i(\omega t - kx))$, that is, the underlying Fourier transform pairs from frequency–wavenumber domain to the space–time domain are of the form

$$ f(\omega, k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t, x)e^{-i(\omega t - kx)}dt \, dx = \mathcal{F}[f(t, x)] $$

$$ f(t, x) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\omega, k)e^{i(\omega t - kx)}d\omega \, dk = \mathcal{F}^{-1}[f(\omega, k)] $$

Important consequences of this transformation convention concern the direction of positive wave propagation and decay, and the location of poles for the dynamic system under consideration. These, in turn, relate to the principles of radiation, boundedness at infinity, and causality. Also, this convention calls for the use of second (cylindrical or spherical) Hankel functions when formulating wave propagation problems in infinite media, either in cylindrical or in spherical coordinates, and casting them in the frequency domain.

1.3 Coordinate systems and differential operators

We choose Cartesian coordinates in three-dimensional space forming a right-handed system, and we denote these indifferently as either $x, y, z$ or $x_1, x_2, x_3$. In most cases, we shall assume that $x = x_1$ and $y = y_2$ lie in a horizontal plane, and that $z = x_3$ is up. For two-dimensional (plane strain) problems, the in-plane (or SV-P) components will be contained in the vertical plane defined by $x$ and $z$ (i.e., $x_1, x_3$), and the anti-plane (or SH) components will be in the horizontal direction $y$ (or $x_2$), which is perpendicular to the plane of wave propagation. On the one hand, this convention facilitates the conversion between Cartesian and either cylindrical or spherical coordinates; on the other, it provides a convenient $x$–$y$ reference system when working in horizontal planes (i.e., in a bird’s-eye view). Nonetheless, you may rotate these systems to suit your convenience.

1.3.1 Cartesian coordinates

a) Three-dimensional space (Fig. 1.1a)

Source–receiver distance

$$ R = \sqrt{x^2 + y^2 + z^2} $$

(1.3a)
1.3 Coordinate systems and differential operators

![Diagram of Cartesian coordinates](image)

Figure 1.1: Cartesian coordinates.

Direction cosines of \( R \)

\[
\gamma_i = \cos \theta_i = \frac{\partial R}{\partial x_i} = \frac{x_i}{R} \tag{1.3b}
\]

First derivatives of direction cosines

\[
\gamma_i, j = \frac{\partial \gamma_i}{\partial x_j} = \frac{1}{R} (\delta_{ij} - \gamma_i \gamma_j) \tag{1.3c}
\]

Second derivatives of direction cosines

\[
\gamma_i, k = \frac{1}{R^2} (3 \gamma_i \gamma_j \gamma_k - \gamma_i \delta_{jk} - \gamma_j \delta_{ki} - \gamma_k \delta_{ij}) \tag{1.3d}
\]

Implied summations

\[
\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3, \quad \gamma_i \gamma_i = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.3e}
\]

Nabla operator

\[
\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \tag{1.4}
\]

Gradient of vector

\[
\nabla \mathbf{u} = \frac{\partial u_x}{\partial x} \hat{i} + \frac{\partial u_y}{\partial y} \hat{j} + \frac{\partial u_z}{\partial z} \hat{k} + \frac{\partial u_x}{\partial y} \hat{i} + \frac{\partial u_y}{\partial z} \hat{j} + \frac{\partial u_z}{\partial z} \hat{k} + \frac{\partial u_x}{\partial z} \hat{i} + \frac{\partial u_y}{\partial z} \hat{j} + \frac{\partial u_z}{\partial z} \hat{k} \tag{1.5}
\]

where the products of the form \( \hat{i} \hat{i} \), etc., are tensor bases, or *dyads*, that is, \( \nabla \mathbf{u} \) is a tensor.

For example, two distinct projections of this tensor are

\[
\hat{k} \cdot \nabla \mathbf{u} = \frac{\partial u_z}{\partial z} \hat{i} + \frac{\partial u_z}{\partial z} \hat{j} + \frac{\partial u_z}{\partial z} \hat{k} = \frac{\partial \mathbf{u}}{\partial z} \tag{1.6}
\]

and

\[
(\nabla \mathbf{u}) \cdot \hat{k} = \frac{\partial u_x}{\partial x} \hat{i} + \frac{\partial u_y}{\partial y} \hat{j} + \frac{\partial u_z}{\partial z} \hat{k} = \nabla u_z \tag{1.7}
\]
Divergence
\[ \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \] (1.8)

Curl
\[ \nabla \times \mathbf{u} = \left( \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \hat{i} + \left( \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \hat{j} + \left( \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{k} \] (1.9)

Curl of curl
\[ \nabla \times \nabla \times \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla \cdot \nabla \mathbf{u} \] (1.10)

Laplacian
\[ \nabla^2 \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \nabla^2 u_x \hat{i} + \nabla^2 u_y \hat{j} + \nabla^2 u_z \hat{k} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{u} \] (1.12)

Wave equation
\[ (\lambda + \mu) \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \mathbf{b} = \rho \ddot{\mathbf{u}} \] (1.13)

b) Two-dimensional space (x–z), (Fig. 1.1b)

Source–receiver distance
\[ r = \sqrt{x^2 + z^2} \] (1.14a)

Direction cosines of \( r \)
\[ \gamma_1 = \cos \theta_1 = \sin \theta_z, \] (1.14b)
\[ \gamma_2 = 0, \] (1.14c)
\[ \gamma_3 = \cos \theta_3 = \cos \theta_z \] (1.14d)

Implied summations
\[ \delta_{ii} = \delta_{11} + \delta_{33} = 2, \] (1.14e)
\[ \gamma_i \gamma_i = \gamma_1^2 + \gamma_3^2 = 1 \] (1.14f)

1.3.2 Cylindrical coordinates

Source–receiver distance
\[ R = \sqrt{x^2 + y^2 + z^2} \] (1.15a)

Range
\[ r = \sqrt{x^2 + y^2} \] (1.15b)

Azimuth
\[ \tan \theta = y/x \] (1.15c)
1.3 Coordinate systems and differential operators

Direction cosines

\[ \gamma_1 = \frac{r \cos \theta}{\sqrt{r^2 + z^2}} = \frac{r \cos \theta}{R} \]  
(1.15d)

\[ \gamma_2 = \frac{r \sin \theta}{\sqrt{r^2 + z^2}} = \frac{r \sin \theta}{R} \]  
(1.15e)

\[ \gamma_3 = \frac{z}{\sqrt{r^2 + z^2}} = \frac{z}{R} \]  
(1.15f)

Basis vectors

\[ \hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta \]  
(1.15g)

\[ \hat{t} = -\hat{i} \sin \theta + \hat{j} \cos \theta \]  
(1.15h)

\[ \hat{k} = \hat{k} \]  
(1.15i)

Conversion between cylindrical and Cartesian coordinates

\[ \mathbf{u} = u_r \hat{r} + u_\theta \hat{t} + u_z \hat{k} \]

\[ = u_r \hat{r} + u_\theta \hat{t} + u_z \hat{k} \]  
(1.16)

\[
\begin{bmatrix}
  u_r \\
  u_\theta \\
  u_z
\end{bmatrix}
= \begin{bmatrix}
  \cos \theta & \sin \theta & 0 \\
  -\sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  u_x \\
  u_y \\
  u_z
\end{bmatrix}
\]  
(1.17)

\[
\begin{bmatrix}
  u_x \\
  u_y \\
  u_z
\end{bmatrix}
= \begin{bmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  u_r \\
  u_\theta \\
  u_z
\end{bmatrix}
\]  
(1.18)

Nabla operator

\[ \nabla = \hat{r} \frac{\partial}{\partial r} + \hat{t} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{k} \frac{\partial}{\partial z} \]  
(1.19)

Allowing the symbol \( \otimes \) to stand for the scalar product, the dot product, or the cross product, and considering that
\[ \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \quad \frac{\partial \hat{k}}{\partial \theta} = 0 \] (1.20)

we can write a generic nabla operation on a vector \( \mathbf{u} \) in cylindrical coordinates as

\[
\nabla \otimes \mathbf{u} = \left( \frac{\partial}{\partial r} \hat{r} + \hat{\theta} - \frac{1}{r} \frac{\partial}{\partial \theta} \hat{r} + \hat{k} \frac{\partial}{\partial z} \right) \otimes \left( \hat{r} u_r + \hat{\theta} u_\theta + \hat{k} u_z \right)
\]

\[
= \frac{\partial u_r}{\partial r} \hat{r} \otimes \hat{r} + \frac{\partial u_\theta}{\partial r} \hat{\theta} \otimes \hat{r} + \frac{\partial u_z}{\partial r} \hat{k} \otimes \hat{r}
\]

\[
+ \frac{1}{r} \left[ \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) \hat{\theta} \otimes \hat{r} + \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) \hat{\theta} \otimes \hat{\theta} + \frac{\partial u_z}{\partial \theta} \hat{k} \otimes \hat{k} \right]
\]

\[
+ \frac{\partial u_r}{\partial z} \hat{r} \otimes \hat{\theta} + \frac{\partial u_\theta}{\partial z} \hat{\theta} \otimes \hat{\theta} + \frac{\partial u_z}{\partial z} \hat{k} \otimes \hat{k}
\] (1.21)

Specializing this expression to the scalar, dot, and cross products, we obtain

**Gradient**

\[
\nabla \mathbf{u} = \frac{\partial u_r}{\partial r} \hat{r} \hat{r} + \frac{\partial u_\theta}{\partial r} \hat{\theta} \hat{r} + \frac{\partial u_z}{\partial r} \hat{k} \hat{r}
\]

\[
+ \frac{1}{r} \left[ \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) \hat{\theta} \hat{r} + \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) \hat{\theta} \hat{\theta} + \frac{\partial u_z}{\partial \theta} \hat{k} \hat{k} \right]
\]

\[
+ \frac{\partial u_r}{\partial z} \hat{r} \hat{\theta} + \frac{\partial u_\theta}{\partial z} \hat{\theta} \hat{\theta} + \frac{\partial u_z}{\partial z} \hat{k} \hat{k}
\] (1.22)

where the products of the form \( \hat{r} \hat{r} \), etc., are tensor bases, or dyads.

**Divergence**

\[
\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) + \frac{\partial u_z}{\partial z} = \frac{1}{r} \left[ \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial (ru_z)}{\partial z} \right]
\] (1.23)

**Curl**

\[
\nabla \times \mathbf{u} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{r} + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \hat{k}
\] (1.24)

**Curl of curl**

\[
\nabla \times \nabla \times \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})
\] (1.25)

\[
\nabla \nabla \cdot \mathbf{u} = \hat{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right] + \hat{\theta} \frac{\partial}{\partial \theta} \left[ \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right]
\]

\[
+ \hat{k} \frac{\partial}{\partial z} \left[ \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right]
\] (1.26)
1.3 Coordinate systems and differential operators

Laplacian
\[ \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]

\[ \nabla^2 u = \nabla \cdot \nabla u = i \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 u_r}{\partial \theta^2} - u_r - 2 \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial^2 u_r}{\partial z^2} \right] \]

\[ + \hat{t} \left[ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 u_\theta}{\partial \theta^2} - u_\theta + 2 \frac{\partial u_\theta}{\partial \theta} \right) \right] \]

\[ + \hat{k} \left[ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right) \right] \]

(Note: \( \partial^2 / \partial \theta^2 \) in \( \nabla^2 \) acts both on the components of \( u \) and on the basis vectors \( \hat{r}, \hat{t}, \hat{k} \).)

Wave equation
\[ (\lambda + \mu) \nabla \nabla \cdot u + \mu \nabla \cdot \nabla u + b = \rho \ddot{u} \] (1.28)

Expansion of a vector in Fourier series in the azimuth
\[ u = \sum_{n=0}^{\infty} u_n \left( \cos n\theta \sin n\theta \right) \] (1.29a)

\[ v = \sum_{n=0}^{\infty} v_n \left( -\sin n\theta \cos n\theta \right) \] (1.29b)

\[ w = \sum_{n=0}^{\infty} w_n \left( \cos n\theta \sin n\theta \right) \] (1.29c)

in which \( u \equiv u_r, v \equiv u_\theta, w \equiv u_z \), and either the lower or the upper element in the parentheses must be used, as may be necessary. Also, \( u_n, v_n, w_n \) are the coefficients of the Fourier series, which do not depend on \( \theta \), but only on \( r \) and \( z \), that is, \( u_n = u_n(r, z) \), and so forth.

1.3.3 Spherical coordinates

Source–receiver distance
\[ R = \sqrt{x^2 + y^2 + z^2} \] (1.30a)

Range
\[ r = \sqrt{x^2 + y^2} \] (1.30b)

Azimuth
\[ \tan \theta = y/x, \quad 0 \leq \theta \leq 2\pi \] (1.30c)

Polar angle
\[ \phi = \arccos \left( \frac{z}{R} \right), \quad 0 \leq \phi \leq \pi \] (1.30d)

Direction cosines
\[ \gamma_1 = \sin \phi \cos \theta \quad \Rightarrow \quad \cos \theta = \frac{\gamma_1}{\sqrt{1 - \gamma_1^2}} \] (1.30e)

\[ \gamma_2 = \sin \phi \sin \theta \quad \Rightarrow \quad \sin \theta = \frac{\gamma_2}{\sqrt{1 - \gamma_2^2}} \] (1.30f)

\[ \gamma_3 = \cos \phi \quad \Rightarrow \quad \sin \phi = \sqrt{1 - \gamma_3^2} \] (1.30g)
Fundamentals

Figure 1.3: Spherical coordinates.

Basis vectors
\[ \hat{r} = \hat{i} \sin \phi \cos \theta + \hat{j} \sin \phi \sin \theta + \hat{k} \cos \phi \] Radial \hspace{1cm} (1.31a)
\[ \hat{i} = -\hat{i} \sin \theta + \hat{j} \cos \theta \] Tangential \hspace{1cm} (1.31b)
\[ \hat{s} = \hat{i} \cos \phi \cos \theta + \hat{j} \cos \phi \sin \theta - \hat{k} \sin \phi \] Meridional \hspace{1cm} (1.31c)

Conversion between spherical and Cartesian coordinates
\[ \mathbf{u} = u_r \hat{r} + u_\theta \hat{t} + u_\phi \hat{s} \] (1.32)

\[
\begin{bmatrix}
u_R \\
u_\phi \\
u_\theta
\end{bmatrix}
= \begin{bmatrix}
\sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\
\cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\
-\sin \theta & \cos \theta & 0
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix}
\] (1.33)

\[
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix}
= \begin{bmatrix}
\sin \phi \cos \theta & \cos \phi \cos \theta & -\sin \theta \\
\sin \phi \sin \theta & \cos \phi \sin \theta & \cos \theta \\
\cos \phi & -\sin \phi & 0
\end{bmatrix}
\begin{bmatrix}
u_R \\
u_\phi \\
u_\theta
\end{bmatrix}
\] (1.34)

Nabla operator
\[ \nabla = \hat{r} \frac{\partial}{\partial R} + \hat{s} \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{t} \frac{1}{R \sin \phi} \frac{\partial}{\partial \theta} \] (1.35)

Allowing the symbol \( \otimes \) to stand for the scalar product, the dot product, or the cross product, and considering that
\[ \frac{\partial \hat{r}}{\partial \phi} = \hat{s}, \quad \frac{\partial \hat{s}}{\partial \phi} = -\hat{r}, \quad \frac{\partial \hat{t}}{\partial \phi} = 0, \] (1.36a)
\[ \frac{\partial \hat{r}}{\partial \theta} = \sin \phi \hat{t}, \quad \frac{\partial \hat{s}}{\partial \theta} = \cos \phi \hat{t}, \quad \frac{\partial \hat{t}}{\partial \theta} = -(\sin \phi \hat{r} + \cos \phi \hat{s}) \] (1.36b)