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PART I

Preference and demand

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CHAPTER 1

Preference and rational choice in the theory of consumption

The purpose of this paper is to study the logical foundations of the theory of consumer behavior. Preference relations, in terms of which the rationality of consumer behavior is postulated, are precisely defined as irreflexive, transitive, monotone, convex, and continuous relations over the set of all conceivable commodity bundles. A demand function associates with prices and incomes those commodity bundles that the consumer chooses subject to budgetary restraints. It will be shown that if a demand function with certain qualitative regularity conditions satisfies Samuelson's Weak Axiom of Consumer Behavior, there exists a preference relation from which the demand function is derived. On the other hand, the demand functions derived from preference relations satisfy Samuelson's Weak Axiom and those regularity conditions other than the Lipschitz condition with respect to income.

1 Introduction

The pure theory of consumer behavior is concerned with the structure of choices of commodity bundles made by a *rational* consumer when he is confronted with various prices and incomes. A large part of the theory is devoted to explaining the contents in which the *rationality* of a consumer's behavior is understood (e.g., Hicks ([7], Chap. 1), Samuelson ([18], Chap. 5), Wold and Juréen ([24], Part 2), and Robertson ([14], Part 1,

From *Proceedings of the First Stanford Symposium on Mathematical Methods in the Social Sciences*, Stanford University Press, 1960, and *Preferences, Utility, and Demand*, edited by J. Chipman, L. Hurwicz, M. Richter, and H. Sonnenschein, Harcourt, Brace, Jovanovich, 1971, pp. 1–17, reprinted with permission.

The author is grateful to Kenneth J. Arrow, David Gale, Paul A. Samuelson, and particularly to Hendrik S. Houthakker and Leonid Hurwicz for their valuable comments and suggestions. The original version contained a number of logical errors which have been pointed out to the author by Professor Hurwicz, and the present version incorporates the lines of proof suggested by him.

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especially pp. 13–20). According to the contributions of Pareto [13], Slutsky [21], Hicks and Allen [9], Wold [23], and others, it is now fairly generally agreed that the rationality of a consumer's behavior may be described by postulating that the consumer has a definite preference over all conceivable commodity bundles and that he chooses those commodity bundles that are optimum with respect to his preference subject to budgetary constraints. The structure of those demand functions that are derived from a consumer's preference relations has been investigated by Slutsky [21], Hicks and Allen [9], Hotelling [10], Hicks ([7], Chap. 2), Samuelson ([18], Chap. 5), and others.

Samuelson's "revealed preference" approach, on the other hand, directly postulates the rationality of a consumer's behavior in terms of demand function. In [16], [17], and ([18], pp. 107–117), he shows that almost all properties of the demand functions derived from preferences are consequences of the so-called Weak Axiom of Consumer Behavior (the term introduced in [20]):

If at the price and income of situation 0 *commodity bundle* x^1 *could be chosen but commodity bundle* x^0 *actually has been chosen* (x^0 *is revealed preferred to* x^1), *then at the price and income of situation* 1 *at which commodity bundle* x^1 *is chosen it is impossible to choose commodity bundle* x^0 (x^1 *is not revealed preferred to* x^0).

The integrability condition[1] or the symmetry of the substitution matrix, however, has not been shown to be implied by the Weak Axiom except in the case of two commodities.[2]

It has been the contribution of Ville [22] and Houthakker [11] to show that a necessary and sufficient condition for a demand function to be derived from a consumer's preference is the so-called Strong Axiom of Consumer Behavior:

If for a finite sequence of commodity bundles $x^0, x^1, \ldots, x^s$ *each commodity bundle* x^t *is revealed preferred to commodity bundle* x^{t+1} ($t = 0, 1, \ldots, s-1$), *then commodity bundle* x^s *is not revealed preferred to commodity bundle* x^0.

Houthakker's Strong Axiom obviously implies Samuelson's Weak Axiom. The existing literature, however, is not sufficient[3] to determine whether or not Samuelson's Weak Axiom is strong enough to imply Houthakker's Strong Axiom. In a recent paper [2], Arrow investigates the problem in a general theory of demand and shows that for those demand

[1] For the integrability problem the reader may consult Georgescu–Roegen [6], Samuelson [20], or Wold and Juréen [24].

[2] For the two-commodity case it has been shown by Samuelson [19] that the Weak Axiom implies the Strong Axiom. See also Rose [15].

[3] See Arrow ([2], p. 121, especially footnote 4). It may be noted that an example given in Hicks ([8], pp. 110–111) is not sufficient to deny the statement.

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functions whose domain of definition contains every set consisting of a finite number of commodity bundles, the Weak Axiom implies the Strong Axiom. The demand functions that appear in consumption theory, however, are not defined for sets of a finite number of commodity bundles. The main purpose of this chapter is to prove that for demand functions with certain qualitative regularity conditions, Samuelson's Weak Axiom implies Houthakker's Strong Axiom. The whole theory of rational consumer behavior may, therefore, be built on the basis of Samuelson's Weak Axiom of Consumer Behavior.

In Section 2 we are concerned with giving a rigorous definition for a consumer's demand functions. A consumer's demand function $h(p, M)$ is a function that is defined for any positive price vector $p = (p_1, p_2, \ldots, p_n)$ and positive income M and has a value in the set of conceivable commodity bundles. It will be supposed furthermore that any positive commodity bundle may be chosen under suitable prices and incomes, and that the total expenditure is equal to income M.

In Section 3 a system of axioms for a consumer's preference relation will be postulated. A preference relation is defined as a relation over the set of all conceivable commodity bundles that is irreflexive, transitive, monotone, convex, and continuous. A demand function will be derived by associating with prices and incomes those commodity bundles that are optimum under a given preference relation in the budget sets.

In Section 4 we show that if a demand function satisfies the Strong Axiom of Consumer Behavior, it is possible to find a preference relation P^* from which the original demand function is derived. It is also noted that the preference relation P^* is uniquely determined by the demand function $h(p, M)$. The relation P^* is generated by successive applications of revealed preference relations.

In order to find a condition under which the Strong Axiom is implied by the Weak Axiom, we introduce the function $M^b = \rho_{b,a}(M^a)$, which maps an income M^a at price p^a into an income M^b at price p^b. It will be shown in Section 5 that a demand function satisfies the Strong Axiom if and only if the Weak Axiom holds and the function $\rho_{b,a}(M^a)$ is strictly increasing.

In Section 6 we shall be concerned with the Pareto-Slutsky-Hicks-Allen approach to the theory of consumer choice. It will be shown that if there is a consumer's preference relation P, then a demand function is uniquely determined and satisfies the Strong Axiom of Consumer Behavior and the regularity conditions other than the Lipschitz condition with respect to income. The preference relation P^*, generated by the derived demand function, coincides with the original preference relation P.

Finally, in Section 7 it will be noted that the symmetry of the substitution matrix may be proved directly in terms of demand functions.

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2 Demand functions

We are concerned with an individual consumer or a group of consumers faced with the problem of choosing a commodity bundle subject to market prices and incomes. Suppose there are n commodities, labeled $i = 1, 2, \ldots, n$, and the individual consumer under consideration may conceivably choose every commodity i in any amount not less than a given quantity x_i^0. A *commodity bundle* $x = (x_1, x_2, \ldots, x_n)$ is an n-vector whose ith component x_i specifies the amount of commodity i to be consumed. The set of all conceivable commodity bundles will be denoted by Ω. By changing the origin of the scale, if necessary, we may without loss of generality assume that minimum available quantities are all zero: $x_i^0 = 0 (i = 1, 2, \ldots, n)$. Now Ω is the set of all nonnegative n-vectors:

$$\Omega = \{x = (x_1, x_2, \ldots, x_n) : x \geqq 0\}.^4$$

Market prices of commodities are described by a *price vector* $p = (p_1, p_2, \ldots, p_n)$, where p_i denotes the price of commodity i $(i = 1, 2, \ldots, n)$. We assume throughout the paper that *price vectors have all positive components*. For a given price vector $p = (p_1, p_2, \ldots, p_n)$ and a given income M, the *budget set* $X(p, M)$ is the set of all commodity bundles whose market values evaluated at p do not exceed income M:

$$X(p, M) = \{x = (x_1, x_2, \ldots, x_n) : x \in \Omega, px \leqq M\}.$$

The demand function of the consumer describes the relation of price vectors and incomes to those commodity bundles that the consumer chooses. Precisely, a function $x = h(p, M) = (h_1(p, M), h_2(p, M), \ldots, h_n(p, M))$ is defined as a *demand function* if the following conditions are met:

D.I *The vector $x = h(p, M)$ is a commodity bundle in Ω defined for any price vector $p = (p_1, p_2, \ldots, p_n)$ and positive income M;*

D.II *Any positive commodity bundle $x = (x_1, x_2, \ldots, x_n)$ is chosen for a*

[4] We use the conventional vector notations: for any two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$,

$x \geqq y$ means $x_i \geqq y_i \quad (i = 1, 2, \ldots, n)$,
$x \geqq y$ means $x \geqq y$ but $x \neq y$,
$x > y$ means $x_i > y_i \quad (i = 1, 2, \ldots, n)$,

$$xy = \sum_{i=1}^{n} x_i y_i,$$

and $\|x\|$ stands for the length of vector $x = (x_1, x_2, \ldots, x_n)$:

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

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suitable price vector $p=(p_1,p_2,\ldots,p_n)$ *and suitable income* M: $x=h(p,M)$;

D.III *The commodity bundle* $x=h(p,M)$ *satisfies the budget equation* $ph(p,M)=M$ *for all* $p>0$ *and* $M>0$.

In what follows, the condition D.II will sometimes be replaced by the following more stringent condition:

D.II′ *For any positive commodity bundle* $x=(x_1,x_2,\ldots,x_n)$ *the price vector* $p=(p_1,p_2,\ldots,p_n)$, *at which commodity bundle* x *is chosen, exists and is determined uniquely except for a scalar multiple.*

We need furthermore to assume that demand functions satisfy certain regularity conditions. In particular, we assume

D.IV *The demand function* $h(p,M)$ *satisfies the Lipschitz condition*[5] *with respect to positive income* M.

The condition D.IV implies that $h(p,M)$ is continuous with respect to M.

The Lipschitz condition is satisfied, for example, if the function $h(p,M)$ has a bounded derivative $\partial h_i/\partial M$ with respect to income M. For demand functions, the Lipschitz condition is also satisfied at those price vectors p and incomes M at which no commodity is inferior. In fact, the budget relation D.III implies that for any variation ΔM in income,

$$\sum_{i=1}^{n} p_i \Delta x_i = \Delta M,$$

where $\Delta x_i = h_i(p, M+\Delta M) - h_i(p,M)$ $(i=1,2,\ldots,n)$. If there is no inferior commodity at (p,M) we have $\Delta x_i/\Delta M \geqq 0$ $(i=1,\ldots,n)$ for sufficiently small income variation ΔM. Hence we have $\Delta x_i/\Delta M \leqq 1/p_i$ $(i=1,2,\ldots,n)$.

A commodity bundle x^0 is defined as *revealed preferred* to a commodity bundle x^1 if

$$p^0x^0 \geqq p^0x^1, \quad x^0 \neq x^1, \tag{1}$$

where $x^0=h(p^0,M^0)$ and $x^1=h(p^1,M^1)$ for suitable p^0,p^1 and M^0,M^1. The relation (1) will be denoted by x^0Rx^1.

It is noted that, by D.II,

For any positive commodity bundles $x, y \in \Omega$, $x \geq y$ implies xRy. (2)

[5] A function $h(p,M)$ is said to satisfy the Lipschitz condition with respect to M at (p^0,M^0) if there exist real numbers $\varepsilon>0$ and K such that, for all positive p with $\|p-p^0\|<\varepsilon$ and all nonnegative M', M'' with $|M'-M^0|<\varepsilon$ and $|M''-M^0|<\varepsilon$, we have

$$\|h(p,M')-h(p,M'')\| \leqq K|M'-M''|.$$

The Lipschitz condition with respect to M is said to be satisfied if it is satisfied at all (p^0,M^0).

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A commodity bundle x is defined as *indirectly revealed preferred* to a commodity bundle y if there is a finite set of commodity bundles $x^1, x^2, \ldots, x^s$ such that

$$xRx^1, x^1Rx^2, \ldots, x^sRy. \tag{3}$$

The relation (3) will be denoted by xR^*y. The relation R^* may be called the *indirect revealed preference relation* generated by the demand function $h(p, M)$.

It is easily seen that the relation R^* is transitive:

$$xR^*y \text{ and } yR^*z \text{ imply } xR^*z. \tag{4}$$

To postulate the regularity condition required to investigate the structure of the demand functions derived from preference relations, we introduce a function $M^b = \rho_{b,a}(M^a)$ associated with two arbitrary price vectors p^a and p^b. For any positive income M^a at price p^a, the income $M^b = \rho_{b,a}(M^a)$ is defined as the supremum of those incomes M at price p^b such that the commodity bundle $x^a = h(p^a, M^a)$ is indirectly revealed preferred to the corresponding commodity bundles $x = h(p^b, M)$. In symbols,

$$\rho_{b,a}(M^a) = \sup\{M : h(p^a, M^a)R^*h(p^b, M)\}. \tag{5}$$

The function $\rho_{b,a}(M^a)$ is always a nondecreasing function of M^a. The regularity condition will be formulated as follows:

(R) *For any price vectors p^a and p^b the function $\rho_{b,a}(M^a)$ is strictly increasing.*

Let the function $\rho'_{b,a}(M^a)$ be defined by

$$\rho'_{b,a}(M^a) = \inf\{M : h(p^b, M)R^*h(p^a, M^a)\}. \tag{6}$$

Then, as can be shown, for a demand function satisfying D.I–D.IV and the Weak Axiom (W), the regularity condition (R) is equivalent to the following:

(R)′ *For any price vectors p^a and p^b the function $\rho_{b,a}(M^a)$ is finite and the function $\rho'_{b,a}(M^a)$ is continuous.*

By using the revealed preference relation R, we can state Samuelson's Weak Axiom of Consumer Behavior as

(W) x^0Rx^1 *implies* $\overline{x^1Rx^0}$.

Houthakker's Strong Axiom of Consumer Behavior, on the other hand, may be formulated in terms of the derived relation R^* by

(S) xR^*y *implies* $\overline{yR^*x}$.

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3 Preference relations

It is customary in modern theory of consumer choice to assume that a rational consumer has a preference relation on the set of all conceivable commodity bundles and that he chooses those commodity bundles that are optimum with respect to his preference. In this section, preference relations on the set of commodity bundles are rigorously defined in terms of a system of axioms.[6]

A relation[7] P defined on the set Ω of all conceivable commodity bundles will be called a *preference relation* if the following axioms are satisfied:

P.I *Irreflexivity: For any* $x \in \Omega$, *we have* $\overline{xPx}$.

P.II *Transitivity: For any* $x, y, z \in \Omega$, *the relations* xPy *and* yPz *imply* xPz.

P.III *Monotonicity: For any* $x, y \in \Omega$ *such that* $x \geq y$, *we have* xPy.

P.IV *Convexity:*[8] *For any* $x, y \in \Omega$ *such that* $x \neq y$ *and* xPy, *we have* $(1-\lambda)x + \lambda yPx$ *for all* $0 < \lambda < 1$.

P.V *Continuity: For any* $x^0 \in \Omega$ *the set* $\{x : x \in \Omega, x^0 Px\}$ *is an open set in* Ω.

A preference relation on the set Ω_0 of all positive commodity bundles is similarly defined.[9]

Axiom P.I, in view of P.II, may be replaced by

P.I′ *Asymmetry: For any* $x, y \in \Omega$, *the relation* xPy *implies* $\overline{yPx}$.

Note that Continuity Axiom P.V introduced above is weaker than the usual continuity condition, which requires in addition to P.V the following:

P.VI *For any* $x^0 \in \Omega$, *the set* $\{x : x \in \Omega, xPx^0\}$ *is on open set in* Ω.

Axioms P.V and P.VI are, respectively, equivalent to

P.V′ *For any* $x^0 \in \Omega$, *the set* $\{x : x \in \Omega, \overline{x^0 Px}\}$ *is a closed set.*

P.VI′ *For any* $x^0 \in \Omega$, *the set* $\{x : x \in \Omega, \overline{xPx^0}\}$ *is a closed set.*

If a preference relation P satisfies Axioms P.I–P.VI, then there exists a continuous function $u(x)$ defined on Ω such that

$$\text{For any } x, y \in \Omega,\ xPy \text{ if and only if } u(x) > u(y). \tag{7}$$

[6] The techniques of symbolic logic in terms of which preference relations are defined were first, in economic analysis, employed extensively by Arrow [1].

[7] By a relation P is meant here a binary relation; i.e., for any two elements x and y in Ω, either xPy or $\overline{xPy}$, where $\overline{xPy}$ means the negation of xPy.

[8] Convexity Axiom P.IV in the present form was suggested by Houthakker.

[9] Any preference relation P on Ω induces a preference relation on Ω_0. However, there exists a preference relation on Ω_0 of which it is impossible to find an extension on Ω. For example, let the relation P on Ω_0 be defined by xPy if and only if $x_1 x_2 \cdots x_n > y_1 y_2 \cdots y_n$. Then P is a preference relation on Ω_0 but it is impossible to extend it to Ω. This example is due to Arrow.

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In fact, let the relation P' be defined by $xP'y$, if and only if $\overline{yPx}$. Then it is shown that

$$\text{For any } x, y \in \Omega,\ xP'y \text{ and/or } yP'x. \tag{8}$$

$$\text{For any } x, y, z \in \Omega, \text{ the relations } xP'y \text{ and } yP'z \text{ imply } xP'z. \tag{9}$$

$$\text{For any } x \in \Omega,\ \{y : yP'x\} \text{ and } \{y : xP'y\} \text{ are closed in } \Omega. \tag{10}$$

The relations (8) and (10) are restatements of P.I′ and P.V–P.VI, respectively. In order to prove (9) suppose, to the contrary, that there are commodity bundles x, y, and z so that $xP'y$ and $yP'z$, but $\overline{xP'z}$ $(x \neq y)$. By Axiom P.IV, $(1-\lambda)x + \lambda yPy$ for all $0 < \lambda < 1$. But by Axiom P.V, there is a positive number λ_0 such that $zP(1-\lambda_0)x + \lambda_0 y$. Hence zPy, which is a contradiction.

By applying Debreu's theorem [4, Theorem I, p. 162], therefore, we can find a continuous function $u(x)$ satisfying (7). Axioms P.III and P.IV imply that $u(x)$ is monotone and strictly quasi-concave. On the other hand, let $u(x)$ be any continuous, monotone, strictly quasi-concave function defined on Ω. Then the relation P defined by (7) is a preference relation satisfying Axioms P.I–P.VI.

Let p be a price vector and M an income. A commodity bundle x^0 is called *optimum* with respect to a preference relation P in the budget set $X(p, M)$ if $x^0 \in X(p, M)$ and $x^0 Px$ for all commodity bundles $x \in X(p, M)$, $x \neq x^0$.

A demand function $h(p, M)$ is defined as *derived* from a preference relation P if for every price vector p and positive income M the commodity bundle $h(p, M)$ is optimum with respect to P in the budget set $X(p, M)$.

4 Preference relations generated by demand functions

Theorem 1. Let $h(p, M)$ be a demand function satisfying D.I–D.IV and the Strong Axiom (S). Then the indirect revealed preference relation R^*, generated by $h(p, M)$, is a preference relation on the set Ω_0 of all positive commodity bundles (i.e., R^* satisfies Axioms P.I–P.V) and the demand function $h(p, M)$ is derived from R^*.

Any preference relation from which the demand function $h(p, M)$ is derived coincides with R^* on Ω_0.

Proof. As is noted in (2) and (4), the relation R^* satisfies P.II and P.III. P.I′ is identical with the Strong Axiom (S). We shall prove here that P.IV and P.V hold for the relation P^*.

In order to prove P.IV, we need the following:

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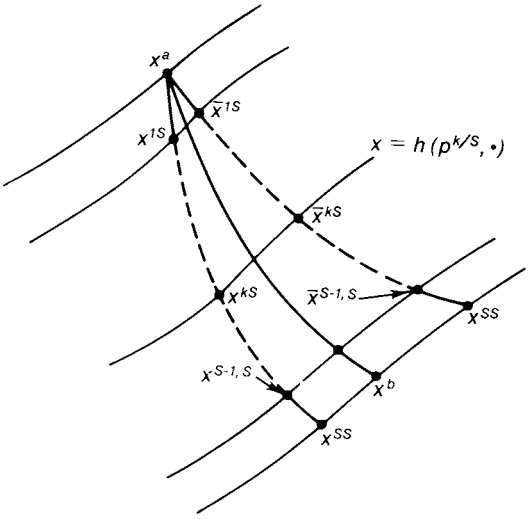

Figure 1.1

Lemma 1. Let the demand function $h(p, M)$ satisfy D.I–D.IV and the Strong Axiom (S). Then, for any price vectors p^a and p^b, we have

$$h(p^a, M^a)R^*h(p^b, M) \quad \text{for all } M < \rho_{b,a}(M^a), \tag{11}$$

and

$$h(p^b, M)R^*h(p^a, M^a) \quad \text{for all } M > \rho_{b,a}(M^a), \tag{12}$$

where $\rho_{b,a}(M^a)$ is defined by (5).

Proof.[10] It may be first noted that the conclusions, (11) and (12), of Lemma 1 are established if the ρ- and ρ'-functions defined by (5) and (6) are identical:

$$\rho_{b,a}(M^a) = \rho'_{b,a}(M^a) \quad \text{for all positive } M^a. \tag{13}$$

The relation (13) will be shown by using the upper and lower income sequences originally introduced by Houthakker [11]. For any positive integer S, let us define the sequence of incomes $\bar{M}^{0S}, \bar{M}^{1S}, \ldots, \bar{M}^{SS}$ recursively by (Figure 1.1).

$$\bar{M}^{k+1,S} = p^{(k+1)/S}\bar{x}^{kS}, \bar{x}^{kS} = h(p^{k/S}, \bar{M}^{kS}), \quad k = 0, 1, \ldots, S-1,$$
$$\bar{M}^{0S} = M^a, \bar{x}^{0S} = x^a = h(p^a, M^a), \tag{14}$$

[10] The present proof of Lemma 1 is based on a slight modification of the technique used by Houthakker [11].

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