A TREATISE
ON THE
ANALYTICAL DYNAMICS
OF PARTICLES AND RIGID BODIES

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ANALYTICAL DYNAMICS
OF PARTICLES AND RIGID BODIES

WITH AN INTRODUCTION TO THE
PROBLEM OF THREE BODIES

BY

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With a foreword by Sir William McCrea, FRS

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CONTENTS

CHAPTER V

THE DYNAMICAL SPECIFICATION OF BODIES

CHAPTER VI

THE SOLUBLE PROBLEMS OF RIGID DYNAMICS

CHAPTER VII

THEORY OF VIBRATIONS

CHAPTER VIII

NON-HOLONOMIC SYSTEMS. DISSIPATIVE SYSTEMS

CHAPTER IX

THE PRINCIPLES OF LEAST ACTION AND LEAST CURVATURE

CHAPTER X

HAMILTONIAN SYSTEMS AND THEIR INTEGRAL-INVARIANTS

CHAPTER XI

THE TRANSFORMATION-THEORY OF DYNAMICS

CHAPTER XII

PROPERTIES OF THE INTEGRALS OF DYNAMICAL SYSTEMS

CHAPTER XIII

THE REDUCTION OF THE PROBLEM OF THREE BODIES

CHAPTER XIV

THE THEOREMS OF BRUNS AND POINCARÉ

CHAPTER XV

THE GENERAL THEORY OF ORBITS

CHAPTER XVI

INTEGRATION BY SERIES

FOREWORD

Edmund Taylor Whittaker (1873–1956) was one of the most remarkable mathematical polymaths of modern times. He was a professional mathematician of outstanding scholarship and originality in mathematical analysis and in the mathematics of general relativity and of several other parts of mathematical physics, as well as of classical mechanics. For several years he was a professional astronomer directing an observatory. He then became a celebrated head of a mathematical department. He was a pioneer of the teaching of numerical mathematics and of mathematical statistics. He was a leading historian of mathematical physics.

As a person, Whittaker was without doubt the most influential figure of his time in the mathematical community of the British Isles. This was partly because of the singular sequence of his appointments. As a young lecturer in Cambridge from 1896 to 1906 he had as colleagues or pupils nearly all the leading British mathematicians of two generations. As Royal Astronomer of Ireland from 1906 to 1912, helped by his temperamental affinity, he gained intimate knowledge of the Irish academic world at what proved to be for it a notable vintage period. Then in Edinburgh from 1912 to 1946 he directed what became the leading *individual* British school of mathematics – those in London, Oxford and Cambridge being more fragmented; there a high proportion of all British mathematicians came under his influence in one capacity or another. Also he became personally acquainted with leading mathematicians throughout the world. He took a lively and considerate interest in everyone he met.

Besides all this Whittaker had a genius for writing monographs and textbooks – and for choosing topics for them. Here, as in everything else, he covered a range that no one else could attempt – books on *Mathematical Analysis, Optical Instruments, Calculus of Observations*, two great volumes of *A History of Theories of Aether and Electricity*, several published lectures or lecture series, and this work on *Analytical Dynamics*.

Another eminent practitioner, E.A. Milne (1896–1950) used to declare that no one should profess to be a mathematical physicist unless he could demonstrate his competence in this field of classical dynamics. For here was a mathematical model, i.e. a body of theorems derived from perfectly formulated postulates. And Milne would have any aspiring mathematical physicist trained first to work strictly within such a discipline, and then to aim to have his own work follow such a pattern. The postulates and the problems are suggested by the world of experience, and there is vital interest in comparing the deductions with further experience. But the model, or theory, is a purely logical mathematical construct. Within the model there can be no doubt about what is right or wrong – it is entirely a matter of doing the mathematics correctly.

Here then we have a classic treatise on this model which we know as 'Newtonian mechanics', and incidentally we are called upon to consider it in the tercentenary year of the first publication of Newton's own account in his *Principia*.

The subject was an abiding interest of Whittaker; this must have become known as far back as 1898, for that year the Council of the British Association resolved 'that Mr E.T. Whittaker be requested to draw up a report on the planetary theory'. This duly appeared in *British Association Report* 1899 pages 121–159 as 'Report on the progress of the solution of the Problem of Three Bodies'. Then in 1912, also necessarily by invitation, he contributed an article on developments of the same topic entitled 'Prinzipien der Störungstheorie' to the famous *Encyklopädie d. math. Wiss. VI.* 2.A pages 512–556, in its section on celestial mechanics.

The four editions of *Analytical Dynamics* appeared in 1904, 1917, 1927, 1937 and the first American printing in 1944. In each edition after the first, Whittaker added references to work published since the preceding one. In particular, although the book remained a treatise on classical mechanics, Whittaker naturally thought it proper to refer to some of the notions and procedures of relativity theory as they came along.

In the early 1920s, there was a phase when the post-World War I mathematicians tended to regard the treatments by Whittaker in *Modern Analysis* and *Analytical Dynamics* as just not in fashion for the up-to-date young man. Then, however, about the year 1925 the most up-to-date mathematical physicists suddenly discovered that *Modern Analysis* provided exactly the mathematics needed for the solution of problems in quantum theory by the methods of 'wave mechanics', and that *Analytical Dynamics* in its transformation theory of dynamics provided exactly the basis needed for the procedures of 'quantum mechanics'.

Whittaker had rightly judged that the 'special functions' of complex analysis, besides possessing mathematical charms, were bound to be of vital interest in connexion with the sort of differential equations cast up by many theories in mathematical physics. So a comprehensive coherent account was certainly needed – hence *Modern Analysis*. Again much of the physical world must behave in a manner that can be described in some generalized sense as 'mechanical' or 'dynamical'. Hence a sufficiently general formulation of the mathematics of dynamical systems was certain to be a continuing requirement. Some such insights – no doubt partly intuitive – seem to have inspired Whittaker in his choice of themes. But in 1925 it looked as though he had had supernatural prescience!

Things have kept on in some such ways. Ever since modern computing resources have been available there has been recurring interest in those problems of classical mechanics that in practice can be usefully solved only by numerical methods. Actually this may be regarded as doubly a tribute to Whittaker's foresight. For not only did he help to transmit the necessary theory by this book, but he has to be credited with having pioneered the university teaching of numerical mathematics. Incredible as it now seems, he instituted a 'mathematical laboratory' for this purpose as part of this department in Edinburgh as far back as 1913. So it would not have surprised him to see his interests in these two fields coming together as fruitfully as they have done in modern times. It was the successful flight of the first sputnik in 1957, just a year after Whittaker's death, that seemed to send all the world scuttling back to its textbooks of classical dynamics and celestial mechanics. And one sequel to all that has been the discoveries in space missions of satellite-ring systems round the outer planets that seem to offer such fruitful scope for the application of the three-body and related problems to which Whittaker devoted so much attention.

Here it has to be noticed that there is a great new interest in classical mechanics that Whittaker – and probably anybody else – cannot be said to have foreseen: the dynamics of *chaos*, or the occurrence of 'determinate but unpredictable' phenomena in dynamical systems. But I think he would have clarified just how it is that the mathematics he had developed leaves room for such phenomena. Also I think that he – and of course Newton too – would have been gleeful over the fact that such an apparently neat and tidy theory should for so very many years have been saving up such surprises for all concerned.

Analytical Dynamics is a classic, a model presentation of a mathematical model. It does not stray outside its well-defined domain. But within the domain it keeps to theorems that are 'interesting' – in the sense explained by G. H. Hardy in *A Mathematician's Apology*. This is in contrast to some treatments that are aridly abstract.

The book serves too as a summary of all the classical work in the field within its scope. It is written as a working monograph, not a history. But the critical footnotes and the references therein summarize the historical development of the subject. They show that Whittaker evidently possessed the material for a history of classical mechanics that would have matched his two-volume work, already mentioned, on most of the rest of classical natural philosophy. Specialists in the field seem in fact to be convinced that *Analytical Dynamics* does contain mention of *all* significant work in the field up to the time of the last edition.

Whittaker obviously had an almost unique talent for grasping other mathematicians' work and seeing precisely how any item fitted into the evolution of its subject as a whole. He must, too, have had his material extraordinarily well organized. As stated, he brought out editions in 1927 and 1937. One knows that in the intervening years he worked, lectured, and had research students working, in quite other topics, mostly concerned with relativity theory. Yet all the time he must have been watching all the literature for developments relevant to the field of the book. Of course it has perhaps to be admitted that classical mechanics does have a somewhat singular status in this regard. If anyone solves a worthwhile unsolved problem or proves a worthwhile theorem his work thereby becomes a permanent contribution. The same cannot be said of most of the other work that fills today's journals of mathematical physics. Most of it plays some part in advancing its subject but nearly all fairly quickly becomes superseded. Most workers would probably like their subjects to be in this regard like classical mechanics. At any rate they should be stimulated by seeing such a subject presented by such a master of exposition as Whittaker.

The first eight chapters are a fairly exhaustive account of Lagrangian dynamics. Chapter IX is then an account of the formulation of dynamics by means of variational principles. Chapters X–XII are an account of the Hamiltonian development of dynamics, upon which the rest of the book is based. This leads first to the fundamental notions of the transformation theory of dynamics in which the motion of a system in accordance with Hamilton's equations is viewed as the 'gradual self-unfolding of a contact formation' (or 'canonical transformation' in later terminology). Incidentally, the theory by which the conditions for a transformation to be canonical are expressed in terms of 'Poisson brackets' is that which Dirac needed for this formulation of quantum mechanics. Dirac had learned it from this book, but when he realized one weekend that this was so, he had to wait until a library reopened on the Monday in order to see the book and refresh his memory.

Chapters XIII and XIV are the *'introduction to the problem of three bodies'* referred to in the book's subtitle, this being presented as a particular application of Hamiltonian dynamics. Chapter XV, on the general theory of orbits, deals mainly with the motion of a particle in a plane under conservative forces. It includes a good accessible short account of the exact solution (in elliptic functions) of the motion of a particle in the field of a single mass in general relativity (Schwarzschild 'external' space-time). There is also a discussion of motion near the 'Lagrange points' in the restricted problem of three bodies, which has well-known application to the Trojan groups of asteroids. Chapter XVI deals with some of the general concepts of integration by series, to which

Whittaker himself made significant contributions. These later chapters include the work of the two reports of 1899 and 1912 mentioned earlier.

The modern reader should find no difficulty with any of Whittaker's terminology or notation. He will ask what some of the work would look like in more modern guise. It may therefore not be out of place to cite V.I. Arnold, *Mathematical Methods of Classical Mechanics* (trans. K. Vogtmann & A. Weinstein) Springer-Verlag 1978, as a book covering a good deal of the same general ground in modern terminology and notation. However, the manipulation required for particular applications cannot in general be much different from Whittaker's.

This last remark serves also to call attention to the copious 'Miscellaneous Examples' throughout this book. Even the most modern reader may be glad to test himself on some of these.

The modern reader, I suggest, will be interested in this book as part of the history of mathematical physics of this century. He should be glad to see what a classic is like. The number of books on the subject written since this one is legion. Most no doubt have special merits. Few have the hard-to-be-defined, but instinctively-to-be-recognized air of being a classic. This book has long been recognized as a genuine specimen.

Because of its completeness the book must continue to serve as a work of reference in its field up to about 1936 when Whittaker signed the Preface of this edition. It will continue to serve as an amplification for any other reading on the subject.

William McCrea

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Frontmatter
More information

PREFACE TO THE FOURTH EDITION

References to work published since 1927 have been inserted, and some errors corrected. For the detection of these I must thank many correspondents.

E. T. WHITTAKER

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