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Convex Bodies: The Brunn–Minkowski Theory

ROLF SCHNEIDER
University of Freiburg

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Basic convexity

1.1. Convex sets and combinations

A set $A \subset \mathbb{E}$ is *convex* if together with any two points x, y it contains the segment $[x, y]$, thus if

$$(1-\lambda)x + \lambda y \in A \quad \text{for } x, y \in A \quad \text{and } 0 \leqq \lambda \leqq 1.$$

Examples of convex sets are obvious; but observe also that $B_0(z, \rho) \cup A$ is convex if A is an arbitrary subset of the boundary of the open ball $B_0(z, \rho)$. As immediate consequences of the definition we note that intersections of convex sets are convex, affine images and pre-images of convex sets are convex and if A, B are convex, then $A + B$ and λA ($\lambda \in \mathbb{R}$) are convex.

Remark 1.1.1. For $A \subset \mathbb{E}^n$ and $\lambda, \mu > 0$ one trivially has $\lambda A + \mu A \supset (\lambda + \mu)A$. Equality (for all $\lambda, \mu > 0$) holds precisely if A is convex. In fact, if A is convex and $x \in \lambda A + \mu A$, then $x = \lambda a + \mu b$ with $a, b \in A$ and hence

$$x = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b\right) \in (\lambda + \mu)A;$$

thus $\lambda A + \mu A = (\lambda + \mu)A$. The other direction of the assertion is trivial.

A set $A \subset \mathbb{E}^n$ is called a *convex cone* if A is convex and nonempty and if $x \in A$, $\lambda \geqq 0$ implies $\lambda x \in A$. Thus a nonempty set $A \subset \mathbb{E}^n$ is a convex cone if and only if A is closed under addition and under multiplication by non-negative real numbers.

By restricting affine and linear combinations to non-negative coefficients, one obtains the following two fundamental notions. The point $x \in \mathbb{E}^n$ is a *convex combination* of the points $x_1, \ldots, x_k \in \mathbb{E}^n$ if there are numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{k} \lambda_i x_i,\ \lambda_i \geqq 0 \quad (i = 1, \ldots, k),\ \sum_{i=1}^{k} \lambda_i = 1.$$

Similarly, the vector $x \in \mathbb{E}^n$ is a *positive combination* of the vectors $x_1, \ldots, x_k \in \mathbb{E}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i \geqq 0 \quad (i = 1, \ldots, k).$$

For $A \subset \mathbb{E}^n$ the set of all convex combinations (positive combinations) of any finitely many elements of A is called the *convex hull* (*positive hull*) of A and is denoted by $\operatorname{conv} A$ ($\operatorname{pos} A$).

Theorem 1.1.2. *If $A \subset \mathbb{E}^n$ is convex, then* $\operatorname{conv} A = A$. *For an arbitrary set $A \subset \mathbb{E}^n$,* $\operatorname{conv} A$ *is the intersection of all convex subsets of $\mathbb{E}^n$ containing A. If $A, B \subset \mathbb{E}^n$, then* $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$.

Proof. Let A be convex. Trivially, $A \subset \operatorname{conv} A$. By induction we show that A contains all convex combinations of any k points of A. For $k = 2$ this holds by the definition of convexity. Suppose that it holds for $k - 1$ and that $x = \lambda_1 x_1 + \ldots + \lambda_k x_k$ with $x_1, \ldots, x_k \in A$, $\lambda_1 + \ldots + \lambda_k = 1$ and $\lambda_1, \ldots, \lambda_k > 0$, without loss of generality. Then

$$x = (1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i + \lambda_k x_k \in A$$

since

$$\frac{\lambda_i}{1 - \lambda_k} > 0, \quad \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} = 1$$

and hence

$$\sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \in A$$

by hypothesis. This proves $A = \operatorname{conv} A$. For arbitrary $A \subset \mathbb{E}^n$ let $D(A)$ be the intersection of all convex sets $K \subset \mathbb{E}^n$ containing A. Since $A \subset \operatorname{conv} A$ and $\operatorname{conv} A$ is evidently convex, we have $D(A) \subset \operatorname{conv} A$. Each convex K with $A \subset K$ satisfies $\operatorname{conv} A \subset \operatorname{conv} K = K$, hence $\operatorname{conv} A \subset D(A)$, which proves the equality.

Let $A, B \subset \mathbb{E}^n$. Let $x \in \operatorname{conv}(A + B)$, thus

$$x = \sum_{i=1}^{k} \lambda_i (a_i + b_i) \quad \text{with } a_i \in A, \quad b_i \in B, \quad \lambda_i \geqq 0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

and hence $x = \sum \lambda_i a_i + \sum \lambda_i b_i \in \operatorname{conv} A + \operatorname{conv} B$. Let $x \in \operatorname{conv} A + \operatorname{conv} B$, thus

$$x = \sum_i \lambda_i a_i + \sum_j \mu_j b_j$$

with $a_i \in A$, $b_j \in B$, $\lambda_i, \mu_j \geqq 0$, $\sum \lambda_i = \sum \mu_j = 1$. We may write

$$x = \sum_{i,j} \lambda_i \mu_j (a_i + b_j)$$

and deduce that $x \in \operatorname{conv}(A + B)$. ■

An immediate consequence is that $\operatorname{conv}(\operatorname{conv} A) = \operatorname{conv} A$.

Theorem 1.1.3. *If* $A \subset \mathbb{E}^n$ *is a convex cone, then* $\operatorname{pos} A = A$. *For a nonempty set* $A \subset \mathbb{E}^n$, $\operatorname{pos} A$ *is the intersection of all convex cones in* $\mathbb{E}^n$ *containing* A. *If* $A, B \subset \mathbb{E}^n$, *then* $\operatorname{pos}(A + B) = \operatorname{pos} A + \operatorname{pos} B$.

Proof. As above. ■

The following result on the generation of convex hulls is fundamental.

Theorem 1.1.4 (Carathéodory's theorem). *If* $A \subset \mathbb{E}^n$ *and* $x \in \operatorname{conv} A$, *then* x *is a convex combination of affinely independent points of* A. *In particular,* x *is a convex combination of* $n + 1$ *or fewer points of* A.

Proof. The point $x \in \operatorname{conv} A$ has a representation

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } x_i \in A, \quad \lambda_i > 0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

with some $k \in \mathbb{N}$, and we may assume that k is minimal. Suppose that $x_1, \ldots, x_k$ are affinely dependent. Then there are numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We can choose m such that λ_m / α_m is positive and, with this restriction, as small as possible (observe that all λ_i are positive and at least one α_i is positive). In the affine representation

$$x = \sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) x_i$$

all coefficients are non-negative (trivially, if $\alpha_i \leqq 0$, otherwise by the choice of m) and at least one of them is zero. This contradicts the minimality of k. Thus $x_1, \ldots, x_k$ are affinely independent, which implies $k \leqq n + 1$. ■

The convex hull of finitely many points is called a *polytope*. A *k-simplex* is the convex hull of $k + 1$ affinely independent points, and these points are the *vertices* of the simplex. Thus Carathéodory's theorem states that conv A is the union of all simplices with vertices in A.

Another equally simple and important result on convex hulls is the following.

Theorem 1.1.5 (Radon's theorem). *Each set of affinely dependent points (in particular, each set of at least $n + 2$ points) in $\mathbb{E}^n$ can be expressed as the union of two disjoint sets whose convex hulls have a common point.*

Proof. If $x_1, \ldots, x_k$ are affinely dependent, there are numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We may assume, after renumbering, that $\alpha_i > 0$ precisely for $i = 1, \ldots, j$; then $1 \leqq j < k$ (at least one α_i is $\neq 0$, say > 0, but not all α_i are > 0). With

$$\alpha := \alpha_1 + \ldots + \alpha_j = -(\alpha_{j+1} + \ldots + \alpha_k) > 0$$

we obtain

$$x := \sum_{i=1}^{j} \frac{\alpha_i}{\alpha} x_i = \sum_{i=j+1}^{k} \left(-\frac{\alpha_i}{\alpha}\right) x_i$$

and thus $x \in \text{conv}\{x_1, \ldots, x_j) \cap \text{conv}\{x_{j+1}, \ldots, x_k\}$. The assertion follows. ■

From Radon's theorem one easily deduces Helly's theorem, a fundamental and typical result of the combinatorial geometry of convex sets.

Theorem 1.1.6 (Helly's theorem). *Let $A_1, \ldots, A_k \subset \mathbb{E}^n$ be convex sets. If any $n + 1$ of these sets have a common point, then all the sets have a common point.*

Proof. Suppose that $k > n + 1$ (for $k < n + 1$ there is nothing to prove, and for $k = n + 1$ the assertion is trivial) and that the assertion is proved for $k - 1$ convex sets. Then for $i \in \{1, \ldots, k\}$ there exists a point

$$x_i \in A_1 \cap \ldots \cap \check{A}_i \cap \ldots \cap A_k$$

where $\check{A}_i$ indicates that A_i has been deleted. The $k \geqq n + 2$ points $x_1, \ldots, x_k$ are affinely dependent; hence from Radon's theorem we can infer that, after renumbering, there is a point

$$x \in \text{conv}\{x_1, \ldots, x_j\} \cap \text{conv}\{x_{j+1}, \ldots, x_k\}$$

for some $j \in \{1, \dots, k-1\}$. Because $x_1, \dots, x_j \in A_{j+1}, \dots, A_k$ we have

$$x \in \operatorname{conv}\{x_1, \dots, x_j\} \subset A_{j+1} \cap \dots \cap A_k,$$

similarly $x \in \operatorname{conv}\{x_{j+1}, \dots, x_k\} \subset A_1 \cap \dots \cap A_j$. ■

Here is a little example (another one is Theorem 1.3.11) to demonstrate how Helly's theorem can be applied to obtain elegant results of a similar nature:

Theorem 1.1.7. *Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{E}^n$ and let $K \subset \mathbb{E}^n$ be convex. If any $n+1$ elements of $\mathcal{M}$ are intersected by some translate of K, then all elements of $\mathcal{M}$ are intersected by a translate of K.*

Proof. Let $\mathcal{M} = \{A_1, \dots, A_k\}$. To any $n+1$ elements of $\{1, \dots, k\}$, say $1, \dots, n+1$, there are $t \in \mathbb{E}^n$ and $x_i \in A_i \cap (K+t)$, hence $-t \in K - A_i$, for $i = 1, \dots, n+1$. Thus any $n+1$ elements of the family $\{K - A_1, \dots, K - A_k\}$ have nonempty intersection. By Helly's theorem there is a vector $-t \in \mathbb{E}^n$ with $-t \in K - A_i$ and hence $A_i \cap (K+t) \neq \emptyset$ for $i \in \{1, \dots, k\}$. ■

Next we look at the interplay between convexity and topological properties. We start with a simple observation.

Lemma 1.1.8. *Let $A \subset \mathbb{E}^n$ be convex. If $x \in \operatorname{int} A$ and $y \in \operatorname{cl} A$, then $[x, y) \subset \operatorname{int} A$.*

Proof. Let $z = (1-\lambda)y + \lambda x$ with $0 < \lambda < 1$. We have $B(x, \rho) \subset A$ for some $\rho > 0$; put $B(o, \rho) =: U$. First we assume $y \in A$. Let $w \in \lambda U + z$, hence $w = \lambda u + z$ with $u \in U$. Then $x + u \in A$, hence $w = (1-\lambda)y + \lambda(x+u) \in A$. This shows that $\lambda U + z \subset A$ and thus $z \in \operatorname{int} A$.

Now assume merely that $y \in \operatorname{cl} A$. Put $V := [\lambda/(1-\lambda)]U + y$. There is some $a \in A \cap V$. We have $a = [\lambda/(1-\lambda)]u + y$ with $u \in U$ and hence $z = (1-\lambda)a + \lambda(x-u) \in A$. This proves that $[x, y) \subset A$, which together with the first part yields $[x, y) \subset \operatorname{int} A$. ■

Theorem 1.1.9. *If $A \subset \mathbb{E}^n$ is convex, then $\operatorname{int} A$ and $\operatorname{cl} A$ are convex. If $A \subset \mathbb{E}^n$ is open, then $\operatorname{conv} A$ is open.*

Proof. The convexity of $\operatorname{int} A$ follows from Lemma 1.1.8. The convexity of $\operatorname{cl} A$ for convex A and the openness of $\operatorname{conv} A$ for open A are easy exercises. ■

The union of a line and a point not on it shows that the convex hull of a closed set need not be closed. This is different for compact sets, as a first application of Carathéodory's theorem shows.

Theorem 1.1.10. *If* $A \subset \mathbb{E}^n$, *then* $\operatorname{conv}\operatorname{cl} A \subset \operatorname{cl}\operatorname{conv} A$. *If A is bounded, then* $\operatorname{conv}\operatorname{cl} A = \operatorname{cl}\operatorname{conv} A$. *In particular, the convex hull of a compact set is compact.*

Proof. $\operatorname{conv}\operatorname{cl} A \subset \operatorname{cl}\operatorname{conv} A$ is easy to see. Let A be bounded; then

$$\{(\lambda_1, \ldots, \lambda_{n+1}, x_1, \ldots, x_{n+1}) | \lambda_i \geqq 0, x_i \in \operatorname{cl} A, \sum_{i=1}^{n+1} \lambda_i = 1\}$$

is a compact subset of $\mathbb{R}^{n+1} \times (\mathbb{E}^n)^{n+1}$, hence its image under the continuous map

$$(\lambda_1, \ldots, \lambda_{n+1}, x_1, \ldots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} \lambda_i x_i \in \mathbb{E}^n$$

is compact. By Carathéodory's theorem, this image is equal to $\operatorname{conv}\operatorname{cl} A$. Thus $\operatorname{cl}\operatorname{conv} A \subset \operatorname{cl}\operatorname{conv}\operatorname{cl} A = \operatorname{conv}\operatorname{cl} A$. ■

The set $\operatorname{cl}\operatorname{conv} A$, which by Theorem 1.1.9 is convex, is called for short the *closed convex hull* of A. This is also the intersection of all closed convex subsets of $\mathbb{E}^n$ containing A.

To obtain information on the relative interiors of convex hulls, we first consider simplices.

Lemma 1.1.11. *Let* $x_1, \ldots, x_k \in \mathbb{E}^n$ *be affinely independent; let* $S := \operatorname{conv}\{x_1, \ldots, x_k\}$ *and* $x \in \operatorname{aff} S$. *Then* $x \in \operatorname{relint} S$ *if and only if in the unique affine representation*

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \textit{with} \sum_{i=1}^{k} \lambda_i = 1$$

all coefficients λ_i *are positive.*

Proof. Clearly we may assume that $k = n + 1$. The condition is necessary since otherwise, because the representation is unique, an arbitrary neighbourhood of x would contain points not belonging to S. To prove sufficiency, let x be represented as above with all $\lambda_i > 0$. Since $x_1, \ldots, x_{n+1}$ are affinely independent, the vectors $\tau(x_1), \ldots, \tau(x_{n+1})$ (see 'Conventions and notation') form a linear basis of $\mathbb{E}^n \times \mathbb{R}$, and for $y \in \mathbb{E}^n$ the coefficients $\mu_1, \ldots, \mu_{n+1}$ in the affine representation

$$y = \sum_{i=1}^{n+1} \mu_i x_i \quad \text{with} \sum_{i=1}^{n+1} \mu_i = 1$$

(the 'barycentric coordinates' of y) are just the coordinates of $\tau(y)$ with respect to this basis. Since coordinate functions in $\mathbb{E}^{n+1}$ are continuous, the coefficients $\mu_1, \ldots, \mu_{n+1}$ depend continuously on y. Therefore, $\delta > 0$ can be chosen such that $\mu_i > 0$ $(i = 1, \ldots, n+1)$ and thus $y \in S$ for all y with $|y - x| < \delta$. This proves $x \in \operatorname{int} S$. ∎

Theorem 1.1.12. *If $A \subset \mathbb{E}^n$ is convex and nonempty, then* $\operatorname{relint} A \neq \emptyset$.

Proof. Let $\dim \operatorname{aff} A = k$, then there are $k+1$ affinely independent points in A. Their convex hull S satisfies $\operatorname{relint} S \neq \emptyset$ by Lemma 1.1.11, furthermore $S \subset A$ and $\operatorname{aff} S = \operatorname{aff} A$. ∎

In view of this theorem, it makes sense to define the *dimension*, $\dim A$, of a convex set A as the dimension of its affine hull. The points of $\operatorname{relint} A$ are also called *internal* points of A.

The description of $\operatorname{relint} \operatorname{conv} A$ for an affinely independent set A given by Lemma 1.1.11 can be extended to arbitrary finite sets.

Theorem 1.1.13. *Let $x_1, \ldots, x_k \in \mathbb{E}^n$; let $P := \operatorname{conv}\{x_1, \ldots, x_k\}$ and $x \in \mathbb{E}^n$. Then $x \in \operatorname{relint} P$ if and only if x can be represented in the form*

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \textit{with } \lambda_i > 0 \quad (i = 1, \ldots, k), \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Proof. We may clearly assume that $\dim P = n$. Suppose that $x \in \operatorname{int} P$. Put

$$y := \sum_{i=1}^{k} \frac{1}{k} x_i;$$

then $y \in P$. Since $x \in \operatorname{int} P$, we can choose $z \in P$ for which $x \in [y, z)$. There are representations

$$z = \sum_{i=1}^{k} \mu_i x_i \quad \text{with } \mu_i \geqq 0, \quad \sum_{i=1}^{k} \mu_i = 1,$$

$$x = (1 - \lambda) y + \lambda z \quad \text{with} \quad 0 \leqq \lambda < 1,$$

which gives

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i = (1 - \lambda)\frac{1}{k} + \lambda \mu_i > 0, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Vice versa, suppose that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i > 0, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

We may assume that $x_1, \ldots, x_{n+1}$ are affinely independent. Put $\lambda := \lambda_1 + \ldots + \lambda_{n+1}$ and

$$y := \sum_{i=1}^{n+1} \frac{\lambda_i}{\lambda} x_i.$$

Lemma 1.1.11 gives $y \in \operatorname{int} \operatorname{conv} \{x_1, \ldots, x_{n+1}\} \subset \operatorname{int} P$. If $k = n + 1$, then $x = y \in \operatorname{int} P$. Otherwise, put

$$z := \sum_{i=n+1}^{k} \frac{\lambda_i}{1 - \lambda} x_i.$$

Then $z \in P$ and $x \in [y, z) \subset \operatorname{int} P$ by Lemma 1.1.8. ■

Theorem 1.1.14. *Let $A \subset \mathbb{E}^n$ be convex. Then*
(a) $\operatorname{relint} A = \operatorname{relint} \operatorname{cl} A$,
(b) $\operatorname{cl} A = \operatorname{cl} \operatorname{relint} A$,
(c) $\operatorname{relbd} A = \operatorname{relbd} \operatorname{cl} A = \operatorname{relbd} \operatorname{relint} A$.

Proof. We may clearly assume that $\dim A = n$. Part (a): trivially, $\operatorname{int} A \subset \operatorname{int} \operatorname{cl} A$. Let $x \in \operatorname{int} \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. There is $z \in \operatorname{cl} A$ with $x \in [y, z)$, and Lemma 1.1.8 shows that $x \in \operatorname{int} A$. Part (b): trivially, $\operatorname{cl} A \supset \operatorname{cl} \operatorname{int} A$. Let $x \in \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. By Lemma 1.1.8 we have $[y, x) \subset \operatorname{int} A$, hence $x \in \operatorname{cl} \operatorname{int} A$. Part (c): $\operatorname{bd} \operatorname{cl} A = \operatorname{cl} (\operatorname{cl} A) \setminus \operatorname{int} (\operatorname{cl} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (a). Then $\operatorname{bd} \operatorname{int} A = \operatorname{cl} (\operatorname{int} A) \setminus \operatorname{int} (\operatorname{int} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (b). ■

We end this section with a definition of the central notion of this book. A nonempty, compact, convex subset of $\mathbb{E}^n$ is called a *convex body* (thus in our terminology, a convex body need not have interior points). By $\mathcal{K}^n$ we denote the set of all convex bodies in $\mathbb{E}^n$ and by $\mathcal{K}_0^n$ the subset of convex bodies with interior points. For $\emptyset \neq A \subset \mathbb{E}^n$ we write $\mathcal{K}(A)$ for the set of convex bodies contained in A and $\mathcal{K}_0(A) = \mathcal{K}(A) \cap \mathcal{K}_0^n$. Further, $\mathcal{P}^n$ denotes the set of nonempty polytopes in $\mathbb{E}^n$, and $\mathcal{P}_0^n = \mathcal{P}^n \cap \mathcal{K}_0^n$.

Notes for Section 1.1

1. The early history of the theorems of Carathéodory, Radon and Helly, and many generalizations, ramifications and analogues of these theorems forming an essential part of combinatorial convexity can be studied in the survey article of Danzer, Grünbaum & Klee [32], which is still strongly recommended. Various results related to Carathéodory's theorem can be found in Reay (1965). An important extension of Radon's theorem is Tverberg's theorem (Tverberg 1966, 1981): Each set of at least $(m - 1)(n + 1) + 1$ points in $\mathbb{E}^n$ (where $m \geqq 2$) can be partitioned into m subsets whose convex hulls have a common point. A survey of later developments is given by Eckhoff (1979). There one also finds hints about more recent developments of the theorems of Carathéodory, Radon and Helly in the abstract setting of so-called convexity spaces.

2. It is clear how a version of Carathéodory's theorem for convex cones is to be formulated and how it can be proved. A common generalization, a version of Carathéodory's theorem for 'convex hulls of points and directions', is given by Rockafellar [29], Theorem 17.1.

1.2. The metric projection

In this section, $A \subset \mathbb{E}^n$ is a fixed nonempty closed convex set. To each $x \in \mathbb{E}^n$ there exists a unique point $p(A, x) \in A$ satisfying

$$|x - p(A, x)| \leqq |x - y| \quad \text{for all } y \in A.$$

In fact, for suitable $\rho > 0$ the set $B(x, \rho) \cap A$ is compact and nonempty, hence the continuous function $y \mapsto |x - y|$ attains a minimum on this set, say at y_0; then $|x - y_0| \leqq |x - y|$ for all $y \in A$. If, also, $y_1 \in A$ satisfies $|x - y_1| \leqq |x - y|$ for all $y \in A$, then $z =: (y_0 + y_1)/2 \in A$ and $|x - z| < |x - y_0|$, except if $y_0 = y_1$. Thus $y_0 =: p(A, x)$ is unique.

In this way a map $p(A, \cdot): \mathbb{E}^n \to A$ is defined; it is called the *metric projection* or *nearest-point map* of A. It will play an essential role in Chapter 4 when the volume of local parallel sets is investigated. It also provides a simple approach to the basic support and separation properties of convex sets (see the next section), as used by Botts (1942) and McMullen & Shephard [26].

We have $|x - p(A, x)| = d(A, x)$, and for $x \in \mathbb{E}^n \setminus A$ we denote by

$$u(A, x) := \frac{x - p(A, x)}{d(A, x)}$$

the unit vector pointing from the nearest point $p(A, x)$ to x and by

$$R(A, x) := \{p(A, x) + \lambda u(A, x) | \lambda \geqq 0\}$$

the ray through x with endpoint $p(A, x)$.

Lemma 1.2.1. *Let $x \in \mathbb{E}^n \setminus A$ and $y \in R(A, x)$; then $p(A, x) = p(A, y)$.*

Proof. Suppose that $p(A, y) \neq p(A, x)$. If $y \in [x, p(A, x))$, then

$$\begin{aligned} |x - p(A, y)| &\leqq |x - y| + |y - p(A, y)| \\ &< |x - y| + |y - p(A, x)| \\ &= |x - p(A, x)|, \end{aligned}$$

which is a contradiction. If $x \in [y, p(A, x))$, let $q \in [p(A, x), p(A, y)]$ be the point such that the segment $[x, q]$ is parallel to $[y, p(A, y)]$. Then

$$\frac{|x - q|}{|x - p(A, x)|} = \frac{|y - p(A, y)|}{|y - p(A, x)|} < 1,$$

again a contradiction. ∎

Theorem 1.2.2. *The metric projection is contracting, that is,*

$$|p(A, x) - p(A, y)| \leqq |x - y| \quad \textit{for } x, y \in \mathbb{E}^n.$$

Proof. We may assume that $v := p(A, x) - p(A, y) \neq o$. We assert that

$$\langle x - p(A, x), v \rangle \geqq 0. \qquad (*)$$

If this is false, then $x \notin A$ and the ray $R(A, x)$ meets the hyperplanes through $p(A, y)$ that is orthogonal to v in a point z, and we deduce from Lemma 1.2.1 that

$$|z - p(A, y)| < |z - p(A, x)| = |z - p(A, z)|,$$

which is a contradiction. Hence $(*)$ holds. Analogously we get $\langle y - p(A, y), v \rangle \leqq 0$. Thus the segment $[x, y]$ meets the two hyperplanes that are orthogonal to v and that go through $p(A, x)$ and $p(A, y)$ respectively. Now the assertion is obvious. ■

Lemma 1.2.3. *Let S be a sphere containing A in its interior. Then* $p(A, S) = \operatorname{bd} A$.

Proof. $p(A, S) \subset \operatorname{bd} A$ is clear. Let $x \in \operatorname{bd} A$. For $i \in \mathbb{N}$ choose x_i in the interior of S (that is, of the ball bounded by S) such that $x_i \notin A$ and $|x_i - x| < 1/i$. From Theorem 1.2.2 we have

$$|x - p(A, x_i)| = |p(A, x) - p(A, x_i)| \leqq |x - x_i| < 1/i.$$

The ray $R(A, x_i)$ meets S in a point y_i and we have $p(A, y_i) = p(A, x_i)$, hence $|x - p(A, y_i)| < 1/i$. A subsequence $(y_{i_j})_{j \in \mathbb{N}}$ converges to a point $y \in S$. From $\lim p(A, y_i) = x$ and the continuity of the metric projection we see that $x = p(A, y)$. Thus $\operatorname{bd} A \subset p(A, S)$. ■

The existence of a unique nearest-point map is characteristic of convex sets. We prove this result here to complete the picture, although no use will be made of it.

Theorem 1.2.4. *Let $A \subset \mathbb{E}^n$ be a closed set with the property that to each point of $\mathbb{E}^n$ there is a unique nearest point in A. Then A is convex.*

Proof. Suppose A satisfies the assumption but is not convex. Then there are points x, y with $[x, y] \cap A = \{x, y\}$, and one can choose $\rho > 0$ such that the ball $B = B((x + y)/2, \rho)$ satisfies $B \cap A = \emptyset$. By an elementary compactness argument, the family $\mathcal{B}$ of all closed balls B' containing B and satisfying $(\operatorname{int} B') \cap A = \emptyset$ contains a ball C with maximal radius. By this maximality, there is a point $p \in C \cap A$, and by the assumed uniqueness of nearest points in A it is unique. If $\operatorname{bd} B$ and bd C have a common point, let this (unique) point be q, otherwise let q be the centre of B. For sufficiently small $\varepsilon > 0$, the ball $C + \varepsilon(q - p)$ includes B and does not meet A. Hence, the family $\mathscr{B}$ contains an element with greater radius than that of C, a contradiction. ■

Note for Section 1.2

Theorem 1.2.4 was found independently (in a more general form) by Bunt (1934) and Motzkin (1935); it is usually associated with the name of Motzkin. In general, a subset A of a metric space is called a Chebyshev set if for each point of the space there is a unique nearest point in A. There are several results and interesting open problems concerning the convexity of Chebyshev sets in normed linear spaces. For more information, see Valentine [30], Chapter VII, Marti [25], Chapter IX, Vlasov (1973) and §6 of the survey article by Burago & Zalgaller (1978).

1.3. Support and separation

The simplest support and separation properties of convex sets seem intuitively obvious, and they are easy to prove. Nevertheless, their many applications make them a basic tool in convexity.

Let $A \subset \mathbb{E}^n$ be a subset and $H \subset \mathbb{E}^n$ a hyperplane and let H^+, H^- denote the two closed halfspaces bounded by H. We say that H *supports A at x* if $x \in A \cap H$ and either $A \subset H^+$ or $A \subset H^-$. H is a *support plane* of A or *supports A* if H supports A at some point x, which is necessarily a boundary point of A. If $H = H_{u,\alpha}$ supports A and $A \subset H^-_{u,\alpha} = \{y \in \mathbb{E}^n | \langle y, u\rangle \leqq \alpha\}$, then $H^-_{u,\alpha}$ is called a *supporting half-space* of A and u is called an *exterior* or *outer normal vector* of both $H_{u,\alpha}$ and $H^-_{u,\alpha}$. If, moveover, $H_{u,\alpha}$ supports A at x, then u is an *exterior normal vector of A at x*. A flat E *supports A at x* if $x \in A \cap E$ and E lies in some support plane of A.

Lemma 1.3.1. *Let $A \subset \mathbb{E}^n$ be nonempty, convex and closed and let $x \in \mathbb{E}^n \backslash A$. The hyperplane H through $p(A, x)$ orthogonal to $u(A, x)$ supports A.*

Proof. Clearly $H \cap A \neq \emptyset$. Let H^- be the closed halfspace bounded by H that does not contain x. Suppose there exists some $y \in A$ with $y \notin H^-$. Let z be the point in $[p(A, x), y]$ nearest to x. Then $|x - z| < |x - p(A, x)|$, which contradicts the definition of $p(A, x)$ since $z \in A$. This shows that $A \subset H^-$. ■

Theorem 1.3.2. *Let $A \subset \mathbb{E}^n$ be convex and closed. Then through each boundary point of A there is a support plane of A. If $A \neq \emptyset$ is bounded,* *then to each vector $u \in \mathbb{E}^n \setminus \{o\}$ there is a support plane to A with exterior normal vector u.*

Proof. Let $x \in \text{bd}\, A$. First let A be bounded. By Lemma 1.2.3 there is a point $y \in \mathbb{E}^n \setminus A$ such that $x = p(A, y)$. By Lemma 1.3.1 the hyperplane through $p(A, y) = x$ orthogonal to $y - x$ supports K at x.

If A is unbounded, there exists through x a support plane H of $A \cap B(x, 1)$; let H^- be the corresponding supporting halfspace of $A \cap B(x, 1)$. If there is a point $z \in A \setminus H^-$, then $[z, x] \subset A$, but $[z, x) \cap B(x, 1) \not\subset H^-$, a contradiction. Hence H supports A.

Let A be bounded and $u \in \mathbb{E}^n \setminus \{o\}$. Since A is compact, there is a point $x \in K$ satisfying $\langle x, u \rangle = \sup \{\langle y, u \rangle | y \in K\}$. Evidently $\{y \in \mathbb{E}^n | \langle y, u \rangle = \langle x, u \rangle\}$ is a support plane to A with exterior normal vector u. ■

The existence of support planes through arbitrary boundary points is characteristic for convex sets, in the following precise sense:

Theorem 1.3.3. *Let $A \subset \mathbb{E}^n$ be a closed set such that* $\text{int}\, A \neq \emptyset$ *and such that through each boundary point of A there is a support plane to A. Then A is convex.*

Proof. Suppose that A satisfies the assumptions but is not convex. Then there are points $x, y \in A$ and $z \in [x, y]$ with $z \notin A$. Since $\text{int}\, A \neq \emptyset$ (and $n \geqq 2$, as we may clearly assume), we can choose $a \in \text{int}\, A$ such that x, y, a are affinely independent. There is a point $b \in \text{bd}\, A \cap [a, z)$. By assumption, through b there exists a support plane H to A, and $a \notin H$ because $a \in \text{int}\, A$. Hence H intersects the plane $\text{aff}\{x, y, a\}$ in a line. The points x, y, a must lie on the same side of this line, which is obviously a contradiction. ■

We turn to separation. Let $A, B \subset \mathbb{E}^n$ be sets and $H_{u,\alpha} \subset \mathbb{E}^n$ a hyperplane. The hyperplane $H_{u,\alpha}$ *separates* A and B if $A \subset H^-_{u,\alpha}$ and $B \subset H^+_{u,\alpha}$, or vice versa. This separation is said to be *proper* if A and B do not both lie in $H_{u,\alpha}$. The sets A and B are *strictly separated* by $H_{u,\alpha}$ if $A \subset \text{int}\, H^-_{u,\alpha}$ and $B \subset \text{int}\, H^+_{u,\alpha}$, or vice versa, and they are *strongly separated* by $H_{u,\alpha}$ if there is an $\varepsilon > 0$ such that $H_{u,\alpha-\varepsilon}$ and $H_{u,\alpha+\varepsilon}$ both separate A and B. Separation of A and a point x means separation of A and $\{x\}$. We first consider this special case:

Theorem 1.3.4. *Let $A \subset \mathbb{E}^n$ be convex and let $x \in \mathbb{E}^n \setminus A$. Then A and x can be separated. If A is closed, then A and x can be strongly separated.*

Proof. If A is closed, the hyperplane through $p(A, x)$ orthogonal to $u(A, x)$ supports A and hence separates A and x. The parallel hyperplane through $(p(A, x) + x)/2$ strongly separates A and x. If A is not closed and $x \notin \operatorname{cl} A$, then a hyperplane separating $\operatorname{cl} A$ and x a fortiori separates A and x. If $x \in \operatorname{cl} A$, then $x \in \operatorname{bd} \operatorname{cl} A$ by Theorem 1.1.14, and by Theorem 1.3.2 there is a support plane to $\operatorname{cl} A$ through x; it separates A and x. ■

Corollary 1.3.5. *Each nonempty closed convex set in $\mathbb{E}^n$ is the intersection of its supporting halfspaces.*

Separation of pairs of sets can be reduced to separation of a set and a point:

Lemma 1.3.6. *Let $A, B \subset \mathbb{E}^n$ be nonempty subsets. A and B can be separated (strongly separated) if and only if $A - B$ and o can be separated (strongly separated).*

Proof. We consider only strong separation; the other case is analogous (or put $\varepsilon = 0$). Suppose that $H_{u,\alpha}$ strongly separates A and B, say $A \subset H^-_{u,\alpha-\varepsilon}$ and $B \subset H^+_{u,\alpha+\varepsilon}$ for some $\varepsilon > 0$. Let $x \in A - B$; thus $x = a - b$ with $a \in A$, $b \in B$. From $\langle a, u\rangle \leqq \alpha - \varepsilon$ and $\langle b, u\rangle \geqq \alpha + \varepsilon$ we get $\langle x, u\rangle \leqq -2\varepsilon$, so that $A - B$ and o are strongly separated by $H_{u,-\varepsilon}$.

Suppose that $A - B$ and o can be strongly separated. Then there are $u \in \mathbb{E}^n \setminus \{o\}$ and $\varepsilon > 0$ such that $\langle x, u\rangle \leqq -2\varepsilon$ for all $x \in A - B$. Let

$$\alpha := \sup\{\langle a, u\rangle \mid a \in A\},$$
$$\beta := \inf\{\langle b, u\rangle \mid b \in B\}.$$

For $a \in A$, $b \in B$ we have $\langle a, u\rangle - \langle b, u\rangle \leqq -2\varepsilon$, hence $\beta - \alpha \geqq 2\varepsilon$. Thus $H_{u,(\alpha+\beta)/2}$ strongly separates A and B. ■

If $A, B \subset \mathbb{E}^n$ are convex, then $A - B$ is convex. If A is compact and B is closed, then $A - B$ is easily seen to be closed. The condition $o \notin A - B$ is equivalent to $A \cap B = \emptyset$. Hence from Lemma 1.3.6 and Theorem 1.3.4 we deduce:

Theorem 1.3.7. *Let $A, B \subset \mathbb{E}^n$ be nonempty convex sets with $A \cap B = \emptyset$. Then A and B can be separated. If A is compact and B is closed, then A and B can be strongly separated.*

The following examples should be kept in mind. Let

$$A := \{(\xi, \eta) \in \mathbb{E}^2 \mid \xi > 0, \eta \geqq 1/\xi\},$$

$$B := \{(\xi, \eta) \in \mathbb{E}^2 | \xi > 0, \eta \leqq -1/\xi\},$$
$$G := \{(\xi, \eta) \in \mathbb{E}^2 | \eta = 0\}.$$

These are pairwise disjoint, closed, convex subsets of $\mathbb{E}^2$. A and B can be strictly separated (by G), but not strongly. $A - B$ and o cannot be strictly separated. A and G can be separated, but not strictly.

On the other hand, convex sets may be separable even if they are not disjoint. The exact condition is given by the following theorem.

Theorem 1.3.8. *Let $A, B \subset \mathbb{E}^n$ be nonempty convex sets. Then A and B can be properly separated if and only if*

$$\operatorname{relint} A \cap \operatorname{relint} B = \emptyset. \qquad (*)$$

Proof. Suppose that $(*)$ holds. Put $C := \operatorname{relint} A - \operatorname{relint} B$. Then $o \notin C$, and C is convex. By Theorem 1.3.4 there exists a hyperplane $H_{u,0}$ with $C \subset H^-_{u,0}$. Let

$$\beta := \inf\{\langle b, u\rangle | b \in B\},$$

then $B \subset H^+_{u,\beta}$. Suppose there exists a point $a \in A$ with $\langle a, u\rangle > \beta$. By Theorem 1.1.12 there exists a point $z \in \operatorname{relint} A$, and Lemma 1.1.8 states that $[z, a) \subset \operatorname{relint} A$. Hence, there is a point $\bar{a} \in \operatorname{relint} A$ with $\langle \bar{a}, u\rangle > \beta$. There is a point $b \in B$ with $\langle b, u\rangle < \langle \bar{a}, u\rangle$ and then, by a similar argument as before, a point $\bar{b} \in \operatorname{relint} B$ with $\langle \bar{b}, u\rangle < \langle \bar{a}, u\rangle$. Thus $\bar{a} - \bar{b} \in C$ and $\langle \bar{a} - \bar{b}, u\rangle > 0$, a contradiction. This shows that $A \subset H^-_{u,\beta}$. Thus A and B are separated by $H_{u,\beta}$. If $A \cup B$ lies in some hyperplane, then this argument yields a hyperplane relative to $\operatorname{aff}(A \cup B)$ separating A and B, and this can be extended to a hyperplane in $\mathbb{E}^n$ that properly separates A and B.

Vice versa, let H be a hyperplane properly separating A and B, say with $A \subset H^-$ and $B \subset H^+$. Suppose there exists $x \in \operatorname{relint} A \cap \operatorname{relint} B$. Then $x \in H$. Since $A \subset H^-$ and $x \in \operatorname{relint} A$, we must have $A \subset H$, similarly $B \subset H$, a contradiction. Thus $(*)$ holds. ■

Occasionally we shall have to use support and separation of convex cones. For these we have:

Theorem 1.3.9. *Let $C \subset \mathbb{E}^n$ be a closed convex cone. Each support plane of C contains* o. *If $x \in \mathbb{E}^n \setminus C$, then there exists a vector $u \in \mathbb{E}^n$ such that*

$$\langle c, u\rangle \geqq 0 \text{ for all } c \in C \text{ and } \langle x, u\rangle < 0.$$

Proof. Let H be a support plane to C. There is a point $y \in H \cap C$. Then $\lambda y \in C$ for all $\lambda > 0$, which is impossible if $o \notin H$. Hence $o \in H$. The rest is clear from Lemma 1.3.1. ■

We shall now prove two more results in the spirit of the theorems of Carathéodory and Helly. They are treated in this section since the first of them needs support planes in its proof and the second one deals with separation.

Theorem 1.3.10 (Steinitz's theorem). *Let $A \subset \mathbb{E}^n$ and $x \in \operatorname{int}\operatorname{conv} A$. Then $x \in \operatorname{int}\operatorname{conv} A'$ for some subset $A' \subset A$ with at most $2n$ points.*

Proof. The point x lies in the interior of a simplex with vertices in $\operatorname{conv} A$; hence, by Carathéodory's theorem applied to each vertex, $x \in \operatorname{int}\operatorname{conv} B$ for some subset B of A with at most $(n+1)^2$ points. We can choose a line G through x that does not meet the affine hull of any $n-1$ points of B. Let x_1, x_2 be the endpoints of the segment $G \cap \operatorname{conv} B$. By Theorem 1.3.2, through x_j there is a support plane H_j to $\operatorname{conv} B$ $(j = 1, 2)$. Clearly $x_j \in \operatorname{conv}(B \cap H_j)$; hence by Carathéodory's theorem there is a representation

$$x_j = \sum_{i=1}^{n} \lambda_{ji} y_{ji} \quad \text{with } y_{ji} \in B, \ \lambda_{ji} \geqq 0, \ \sum_{i=1}^{n} \lambda_{ji} = 1$$

$(j = 1, 2)$, and here necessarily $\lambda_{ji} > 0$ by the choice of G. With suitable $\lambda \in (0, 1)$ we have

$$x = (1-\lambda)x_1 + \lambda x_2 = \sum_{i=1}^{n} [(1-\lambda)\lambda_{1i} y_{1i} + \lambda\lambda_{2i} y_{2i}]$$
$$\in \operatorname{relint}\operatorname{conv}\{y_{11}, \ldots, y_{1n}, y_{21}, \ldots, y_{2n}\}$$

by Theorem 1.1.13. Here relint can be replaced by int, since by the choice of G the points $y_{11}, \ldots, y_{1n}$ are affinely independent and for at least one k also $y_{11}, \ldots, y_{1n}, y_{2k}$ are affinely independent. ■

Theorem 1.3.11 (Kirchberger's theorem). *Let $A, B \subset \mathbb{E}^n$ be compact sets. If for any subset $M \subset A \cup B$ with at most $n+2$ points the sets $M \cap A$ and $M \cap B$ can be strongly separated, then A and B can be strongly separated.*

Proof. First we assume that A and B are finite sets. For $x \in \mathbb{E}^n$ define (with $\tau(x) := (x, 1)$)

$$H_x^{\pm} := \{v \in \mathbb{E}^n \times \mathbb{R} \mid \pm \langle v, \tau(x)\rangle > 0\}.$$

Let $M \subset A \cup B$ and $\operatorname{card} M \leqq n+2$. By the assumption there exist $u \in \mathbb{E}^n$ and $\alpha \in \mathbb{R}$ such that $\langle u, a\rangle > \alpha$ for $a \in M \cap A$ and $\langle u, b\rangle < \alpha$ for $b \in M \cap B$. Writing $v := (u, -\alpha)$, we see that $\langle v, \tau(a)\rangle = \langle u, a\rangle - \alpha > 0$; thus $v \in H_a^+$ for $a \in M \cap A$. Similarly $v \in H_b^-$ for $b \in M \cap B$. Thus the family $\{H_a^+ \mid a \in A\} \cup \{H_b^- \mid b \in B\}$ of finitely many convex sets in $\mathbb{E}^n \times \mathbb{R}$ has the property that any $n + 2$ or fewer of the sets have nonempty intersection. By Helly's theorem, the intersection of all sets in the family is not empty. Since this intersection is open, it contains an element of the form $v = (u, -\alpha)$ with $u \neq o$. For $a \in A$ we have $v \in H_a^+$, hence $\langle u, a \rangle > \alpha$, and for $b \in B$ similarly $\langle u, b \rangle < \alpha$. Hence A and B, being finite sets, are strongly separated by $H_{u,\alpha}$.

Now let A, B be compact sets satisfying the assumption. Suppose that A and B cannot be strongly separated. Then $\operatorname{conv} A$ and $\operatorname{conv} B$ cannot be strongly separated. By Theorem 1.1.10 these sets are compact and hence by Theorem 1.3.7 they cannot be disjoint. Let $x \in \operatorname{conv} A \cap \operatorname{conv} B$. Then $x \in \operatorname{conv} A' \cap \operatorname{conv} B'$ with finite subsets $A' \subset A$ and $B' \subset B$, which hence cannot be strongly separated. This contradicts the result shown above. ∎

We conclude this section with another application of a separation theorem, which will be useful in the study of Minkowski addition.

Lemma 1.3.12. *Let $A, B \subset \mathbb{E}^n$ be nonempty convex sets. If*

$$x \in \operatorname{relint}(A + B),$$

then x can be represented in the form $x = a + b$ with $a \in \operatorname{relint} A$ and $b \in \operatorname{relint} B$.

Proof. There is a representation $x = y + z$ with $y \in A$ and $z \in B$. We may assume that $x = y = z = o$ and also that $\dim(A + B) = n$. Since $o \in \operatorname{int}(A + B)$, $A + B$ and o cannot be separated. Hence, by Lemma 1.3.6 and Theorem 1.3.8, there is a point

$$a \in \operatorname{relint} A \cap \operatorname{relint}(-B).$$

Then $-a \in \operatorname{relint} B$ and $o = a - a$. ∎

Notes for Section 1.3

1. Separation and support properties of convex sets in finite and infinite dimensions are of fundamental importance in various fields such as functional analysis, optimization, control theory, mathematical economy, and others. For infinite-dimensional separation and support theorems we refer only to Bourbaki [9], Marti [25], Holmes (1975) and Bair & Fourneau [3]; see also the survey article by Klee (1969b).

   A thorough study of several types of separation in $\mathbb{E}^n$ was made by Klee (1968).
2. A stronger version of Theorem 1.3.3 (existence of local support planes) is associated with the name of Tietze. A survey of results of this type is given in the article by Burago & Zalgaller (1978).
3. Historical information on the theorems of Steinitz and Kirchberger can be found in the survey article by Danzer, Grünbaum & Klee [32].

4. *Positive bases*. Let $B \subset \mathbb{E}^n$. Using Theorem 1.1.13 one easily sees that $\operatorname{pos} B = \mathbb{E}^n$ holds if and only if $o \in \operatorname{int}\operatorname{conv} B$. The set B is called a *positive basis* of $\mathbb{E}^n$ if $\operatorname{pos} B = \mathbb{E}^n$ but $\operatorname{pos} B' \neq \mathbb{E}^n$ for each proper subset $B' \subset B$. Thus Steinitz's theorem implies that a positive basis of $\mathbb{E}^n$ contains at most $2n$ vectors. If B is a linear basis of $\mathbb{E}^n$, then $B \cup (-B)$ is a positive basis, and up to multiplication by positive numbers it is only in this way that the maximal number $2n$ can be achieved. Positive bases have been investigated by Davis (1954), McKinney (1962), Bonnice & Klee (1963), Reay (1965) and Shephard (1971).

1.4. Extremal representations

The purpose of this section is to represent a closed convex set as the convex hull of a smaller set, and here the smallest possible sets will be of particular interest. A first candidate for a smaller set with the same convex hull is the relative boundary. Only the obvious trivial cases must be excluded:

Lemma 1.4.1. *If $A \subset \mathbb{E}^n$ is a closed convex set with $A \neq \operatorname{conv}\operatorname{relbd} A$, then A is either a flat or a half-flat.*

Proof. Clearly we may assume that $\dim A = n$. There is a point $x \in \operatorname{int} A$ with $x \notin \operatorname{conv}\operatorname{bd} A$ (since otherwise $A = \operatorname{int} A \cup \operatorname{bd} A = \operatorname{conv}\operatorname{bd} A$). By the separation theorem, 1.3.4, there is a closed halfspace H^- such that $x \in H^-$ and $\operatorname{conv}\operatorname{bd} A \subset H^+$. Each point $y \in \operatorname{int} H^-$ satisfies $[x, y] \cap \operatorname{bd} A = \emptyset$ and hence $y \in \operatorname{int} A$; thus $H^- \subset A$. By the convexity and closedness of A, each translate of H^- with a point of A in its boundary is contained in A. Thus A is either equal to $\mathbb{E}^n$ or is a halfspace. ■

We will exclude the exceptional cases that are the subject of Lemma 1.4.1 by demanding that A be *line-free*, meaning that A does not contain a line. Owing to Lemma 1.4.2 below, this is not a severe restriction. First let $A \subset \mathbb{E}^n$ be a closed convex set. Suppose that A contains a ray $G_{x,u} := \{x + \lambda u \mid \lambda \geqq 0\}$ with $x \in \mathbb{E}^n$ and $u \in \mathbb{E}^n \setminus \{o\}$. Let $y \in A$. Let $z \in G_{y,u}$ and $w \in [x, z)$. The ray through w with endpoint y meets $G_{x,u}$, hence $w \in A$. Thus $[x, z) \subset A$ and hence $z \in A$. This shows that also $G_{y,u} \subset A$. For this reason, it makes sense to define

$$\operatorname{rec} A := \{u \in \mathbb{E}^n \setminus \{o\} \mid G_{x,u} \subset A\} \cup \{o\}$$

where $x \in A$; this set does then not depend upon the choice of x. It is evidently a closed convex cone, called the *recession cone* of A. One may also write

$$\operatorname{rec} A = \{u \in \mathbb{E}^n \mid A + u \subset A\}.$$

Lemma 1.4.2. *Each closed convex set $A \subset \mathbb{E}^n$ can be represented in the form $A = \bar{A} \oplus V$, where V is a linear subspace of $\mathbb{E}^n$ and $\bar{A}$ is a line-free closed convex set in a subspace complementary to V.*

Proof. Assume that A is not line-free. Then

$$V := \operatorname{rec} A \cap (-\operatorname{rec} A)$$

(the *lineality space* of A) is the linear subspace consisting of all vectors that are parallel to some line contained in A. Let U be a linear subspace complementary to V and put $\bar{A} := A \cap U$; then $\bar{A} + V \subset A$. Let $x \in A$. Through x there exists a line $G \subset A$; since it is parallel to V, it meets U in a point y. Then $x = y + (x - y)$ with $y \in \bar{A}$ and $x - y \in V$; hence $x \in \bar{A} + V$. This proves $A = \bar{A} \oplus V$. Clearly $\bar{A}$ is closed, convex and line-free. ■

The representation by convex hulls of minimal sets requires some definitions. Let $A \subset \mathbb{E}^n$ be a convex set. A *face* of A is a convex subset $F \subset A$ such that each segment $[x, y] \subset A$ with $F \cap \operatorname{relint}[x, y] \neq \emptyset$ is contained in F or, equivalently, such that $x, y \in A$ and $(x + y)/2 \in F$ implies $x, y \in F$. If $\{z\}$ is a face of A, then z is called an *extreme point* of A. In other words, z is an extreme point of A if and only if it cannot be written in the form $z = (1 - \lambda)x + \lambda y$ with $x, y \in A$ and $\lambda \in (0, 1)$. The set of all extreme points of A is denoted by $\operatorname{ext} A$. An *extreme ray* of A is a ray that is a face of A. By $\operatorname{extr} A$ we denote the union of the extreme rays of A.

Theorem 1.4.3. *Each line-free closed convex set $A \subset \mathbb{E}^n$ is the convex hull of its extreme points and extreme rays;*

$$A = \operatorname{conv}(\operatorname{ext} A \cup \operatorname{extr} A).$$

Proof. The assertion is clear for $n \leqq 1$. Suppose that $n \geqq 2$, $\dim A = n$ (w.l.o.g.) and the assertion has been proved for convex sets of smaller dimension. By Lemma 1.4.1, $A = \operatorname{conv} \operatorname{bd} A$. By the support theorem, 1.3.2, each point $x \in \operatorname{bd} A$ lies in some support plane H of A. By the induction hypothesis, x lies in the convex hull of the extreme points and extreme rays of $H \cap A$, and it is an immediate consequence of the definition of a face that these are respectively extreme points and extreme rays of A itself. The assertion follows. ■

Corollary 1.4.4. *If $A \subset \mathbb{E}^n$ is a line-free closed convex set then*

$$A = \operatorname{conv} \operatorname{ext} A + \operatorname{rec} A.$$