Notes on logic and set theory

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Preface

This book has its origins in a course of lectures entitled 'Set Theory and Logic' which is given to third-year undergraduates in Cambridge. The Cambridge Mathematical Tripos contains rather little on the foundational aspects of mathematics: this course is (at the time of writing – though there are plans for a change before long) the only opportunity which undergraduates have to learn about the basic ideas of logic and axiomatic set theory in an examinable context, and its aim is therefore to provide a general introduction to these ideas for students who, though they may possess considerable sophistication in other areas of mathematics, have no previous experience of logic beyond a nodding acquaintance with 'naive' set theory, and whose primary interests may well lie in other areas of pure (or even applied) mathematics. Having lectured this course in 1984, 1985 and 1986, I have been struck by the fact that there was no single textbook available covering both logic and set theory at the level of rigour and sophistication appropriate to such a course, and – in the belief that there might be universities other than Cambridge where the need for such a textbook was felt – I conceived the idea of expanding my lecture notes into a publishable text. I am glad to say that this idea was enthusiastically received by Cambridge University Press; I am grateful, in particular, to David Tranah and Martin Gilchrist for their support of the project.

The *raison d'être* of this book, then, is to try to collect within a single pair of covers everything that the well-educated mathematician in the late twentieth century needs to know about the foundations of his subject. Though there has, as indicated above, been some expansion as compared with the original lecture course, it has been kept to a bare minimum: anything which is merely of specialist interest to logicians has (I hope) been excluded. (Briefly, the additions consist of the whole of Chapter 4 on recursion theory – there was, regrettably, not enough time to cover recursive functions in the original course – plus more detailed proofs of one or two major results (such as the Completeness Theorem for the Predicate Calculus) which were merely stated, or given sketch-proofs, in the lectures.)

However, I have refrained from giving the book a title of the *Logic for the Working Mathematician* variety, since such titles are often a cover for compromise: in their anxiety to make material accessible to the general reader, authors have a tendency to skate over details when to do otherwise would involve them in discussion of what might be considered specialist matters. (I should hastily add that the last sentence is not intended as a criticism of any particular book whose title may approximate to that quoted.) In this book I have tried to be completely honest with the reader; I have always sought the simplest and most direct proof, but where there are genuine technical difficulties in the way I have not been afraid to say so (even though, in some cases, it has not been possible within the compass of a book like this to give a full account of the difficulties or of the manner of their resolution). In an effort to prevent the text from becoming too indigestible, I have often relegated the discussion of these difficulties to a remark (sometimes headed 'Note for worriers', or something equally facetious) enclosed within square brackets; the reader who wishes to confine himself to the basic ideas and not bother about the technicalities can usually skip over such bracketed passages without loss of continuity.

The layout of the book is as follows. Chapters 1–3 develop the language and machinery of first-order logic, up to the proof of the Completeness Theorem and some of its consequences. Chapter 4 develops recursion theory from its beginnings up to (but not quite including) the Recursion Theorem. Chapters 5–8 develop Zermelo–Fraenkel set theory, beginning with the axioms and working up to the 'traditional' discussion of ordinal and cardinal arithmetic. Finally, Chapter 9 contains a proof of Gödel's Incompleteness Theorems, followed by a fairly informal discussion of the technology of set-theoretic independence proofs. There are exercises at the end of each chapter (except Chapter 9, where it did not seem appropriate);

these range from mere five-finger exercises, through the verification of minor details which have been omitted from the proofs of theorems in the text, to quite substantial problems whose results might well have been included in the text as propositions, had it not been for the policy of exclusion mentioned earlier. The latter kind (in particular) have been provided with what I hope is a generous supply of hints, enclosed within square brackets.

Although it is clearly preferable that a student should work through the whole book from beginning to end, it is possible to meet the demands of shorter courses by omitting material in various ways. In particular, if it is desired to omit recursion theory (as in the course from which the book developed), then Chapter 4 can be skipped without any loss of understanding of Chapters 5–8; there will inevitably be more difficulty with the Incompleteness Theorems in Chapter 9, but I hope that it would still be possible to understand the discussion of them on an informal level. On the other hand, if one is not particularly interested in covering axiomatic set theory, it should be possible to jump from the end of Chapter 4 – or, even better, from about halfway through Chapter 5 – to the beginning of Chapter 9.

The prerequisites for reading the book are fairly few; in particular, though the opening chapters presuppose some acquaintance with naive set theory, there is no presumption that the reader knows anything about logic. Otherwise, his mathematical knowledge is assumed to be what would be usual for a final-year undergraduate in Britain: some familiarity with subjects such as group theory, ring theory and point-set topology is presumed in some of the examples, but the omission of these examples should not seriously damage one's understanding of the mainstream of the text. Thus, although this is not its primary purpose, the book could be used as a first course in logic for rather more specialist logicians than I have envisaged.

There is one respect, however, in which the book is rather conspicuously aimed at non-specialists: namely, I have not been afraid to shoot a few of the sacred cows of logical tradition for the sake of rendering the exposition more smoothly compatible with mathematical practice. Perhaps the most obvious instance occurs in Chapter 3 where, in order to develop a logical calculus which is both sound and complete for possibly-empty models, I found it necessary to introduce the restriction that quantifiers may only be applied to formulae which actually involve the quantified variable [if $(\forall x)\perp$

were allowed as a formula, then neither of the implications $((\forall x)\bot \Rightarrow (\forall x)\neg(x = x))$ and $((\forall x)\neg(x = x) \Rightarrow (\forall x)\bot)$ would be provable]. I can well imagine that professional logicians will find this hard to swallow (I did myself at first, though I am not by training a logician); but I would ask them to reflect, before they condemn me, on whether anyone *other* than a logician would find this restriction at all odd.

Although, as indicated in the first paragraph, the selection of topics covered in this book does not exactly correspond to that in any previous book known to me, there are a number of previous texts from which I have borrowed ideas concerning the arrangement and presentation of material, and to whose authors I must therefore express my indebtedness – it would be obvious in any case from the text itself. For the logic chapters (1–3), I have taken various ideas from *An Algebraic Introduction to Mathematical Logic* by D. W. Barnes and J. M. Mack (Springer-Verlag, 1975); in particular, the idea of beginning a treatment of first-order logic by studying universal algebra is theirs. In Chapter 4, the influence of N. J. Cutland's book *Computability* (Cambridge University Press, 1980) will be apparent to anyone who knows it well. And in the set theory chapters (5–8), the ordering of material (in particular) owes a good deal to *Sets: Naive, Axiomatic and Applied* by D. van Dalen, H. C. Doets and H. de Swart (Pergamon Press, 1978). [The citation of these three texts has an ulterior motive. It occurs to me that many students, having read the deliberately brief accounts of first-order logic, recursion theory and set theory in this book, will want to pursue at least one of them in greater depth. For this purpose, I should be happy to steer them towards the three books just mentioned.]

But, far more than to any of these authors, I am indebted to the three generations of undergraduates who struggled through the lectures that generated successive approximations to these notes, and to my colleagues who, in tutorials, found themselves obliged to patch up my mistakes and eliminate the misunderstandings I had created. It would be invidious to mention any particular names, but it is undoubtedly true that, but for the feedback I received from them, these notes would contain a great many more errors than they do.

Cambridge, January 1987 P. T. Johnstone