OPTIMIZATION AND STABILITY THEORY
FOR ECONOMIC ANALYSIS
To Marguerite—B.B.

To my parents, Terry and Robert—I.M.D.
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PREFACE

This book presents a coherent and systematic exposition of the mathematical theory of the problems of optimization and stability. Both of these are topics central to economic analysis since the latter is so much concerned with the optimizing behaviour of economic agents and the stability of the interaction processes to which this gives rise. A basic knowledge of optimization and stability theory is therefore essential for understanding and conducting modern economic analysis. The book is designed for use in advanced undergraduate and graduate courses in economic analysis and should, in addition, prove a useful reference work for practising economists.

Although the text deals with fairly advanced material, the mathematical prerequisites are minimised by the inclusion of an integrated mathematical review designed to make the text self-contained and accessible to the reader with only an elementary knowledge of calculus and linear algebra. We strongly urge the reader to peruse the material contained in the mathematical review before proceeding to the main text. Furthermore, Chapter 1 on convexity is to some extent a reference chapter, and can be regarded as an extension to the mathematical review in that it presents certain fundamental properties of convex sets and functions which are used throughout the text. The reader with a basic knowledge of convexity may begin with Chapter 2, where the theory of static optimization is developed. Chapter 4 on comparative statics and duality can be read immediately after Chapter 2 if so desired. Chapter 3, which deals with equilibrium mathematics is reasonably self-contained, and could be omitted on a first reading, although the material contained
PREFACE

therein is widely applied in general equilibrium analysis. Stability theory and dynamic optimization are covered in the remaining chapters, and are best read in order.

Many of the more important theorems in the text are proved, although we often sacrifice elegance in the interests of minimising the mathematical prerequisites. An important feature of the text is the broad range of often extended and detailed economic applications. We believe that one of the best ways of enabling the reader to ‘do’ modern economic analysis is by working through such extended examples.

The index should enable the reader to locate relevant definitions and theorems. Although the notation used in the text is fairly standard, for convenience, there is a glossary to the symbols introduced in the mathematical review.

The text is an outgrowth of our experience in teaching optimization and stability theory in various second and third year courses in the economics degree programmes at the University of Newcastle upon Tyne. Whether our students would recognize it as such is another matter! Nevertheless, we would like to acknowledge their constructive and destructive comments and insights. We wish to thank Harvey Thompson and several referees for their helpful comments and insights. Naturally they are absolved from any remaining inaccuracies in the text. Finally, thanks are due to Francis Brooke and Patrick McCartan for their guidance, encouragement and editorial assistance throughout the process of writing this manuscript.

BRIAN BEAVIS
IAN M. DOBBS

University of Newcastle upon Tyne