INTRODUCTION

In Einstein's theory of gravitation matter and its dynamical interaction are based on the notion of an intrinsic geometric structure of the space-time continuum. The ideal aspiration, the ultimate aim, of the theory is not more and not less than this: A four-dimensional continuum endowed with a certain intrinsic geometric structure, a structure that is subject to certain inherent purely geometrical laws, is to be an adequate model or picture of the 'real world around us in space and time' with all that it contains and including its total behaviour, the display of all events going on in it.

Indeed the conception Einstein put forward in 1915 embraced from the outset (and not only by the numerous subsequent attempts to generalize it) every kind of dynamical interaction, not just gravitation only. That the latter is usually in the foreground of our mind—that we usually call the theory of 1915 a theory of gravitation—is due to two facts. First, its early great successes, the new phenomena it predicted correctly, were deemed to refer essentially to gravitation, though that is, strictly speaking, true only for the precession of the perihelion of Mercury. The deflexion of light rays that pass near the sun is not a purely gravitational phenomenon, it is due to the fact that an electromagnetic field possesses energy and momentum, hence also mass. And also the displacement of spectral lines on the sun and on very dense stars ('white dwarfs') is obviously an interplay between electromagnetic phenomena and gravitation.

At any rate the very foundation of the theory, viz. the basic principle of equivalence of acceleration and a gravitational field, clearly means that there is no room for any kind of 'force' to produce acceleration save gravitation, which however is not to be regarded as a force but resides on the geometry of space-time. Thus in fact, though not always in the wording, the mystic concept of force is wholly abandoned. Any 'agent' whatsoever, producing ostensible accelerations, does so quâ amounting to an energy-momentum tensor and via the gravitational field connected with the latter. The case of 'pure gravitational interaction' is distinguished only by being the simplest of its kind, inasmuch as the energy-momentum- (or matter-) tensor can here be regarded as located in minute specks of matter (the particles or mass-points) and as having a particularly simple form, while, for example, an electrically charged particle is connected with a matter-tensor spread throughout the space around it and of a rather complicated form even when the particle is at rest. This has, of course, the consequence that in such a case we are in patent need of field-laws for the matter-tensor (e.g. for the electromagnetic field), laws that one would also like to conceive as purely geometrical restrictions on the structure of space-time. These laws the theory of 1915 does not yield, except in the simple case of purely gravitational interaction. Here the defect can at least be camouflaged or provisionally supplemented by simple additional assumptions such as: the particle shall keep together, there shall be no negative mass, etc. But in other cases, such as electromagnetism, a further development of the geometrical conceptions about space-time is called for, to yield the field-laws of the matter-tensor in a natural fashion. This was the second reason for looking upon the theory of 1915 as referring to pure gravitation only.

The geometric structure of the space-time model envisaged in the 1915 theory is embodied in the following two principles:

(i) equivalence of all four-dimensional systems of coordinates obtained from any one of them by arbitrary (point-) transformation;

(ii) the continuum has a metrical connexion impressed on it: that is, at every point a certain quadratic form of the coordinate-differentials,

$$g_{ik}\,dx_i\,dx_k,$$

called the 'square of the interval' between the two points in question, has a fundamental meaning, invariant in the aforesaid transformations.

These two principles are of very different standing. The first, the principle of general invariance, incarnates the idea of General Relativity. I will not commit myself to calling it unshakable. One has occasionally tried to generalize it, and it is difficult to say whether quantum physics might not at some time seriously dictate its generalization. However, the principle as it stands appears to be simpler than any generalization we might contemplate, and there seems to be no reason to depart from it at the outset.

On the other hand, to adopt a metrical connexion straight away does not seem to be the simplest way of getting at it eventually, even if nothing more were intended than an exposition of the 1915 theory. The reason is that the conceptions on which this theory hinges (as invariant differentiation, Riemann-Christoffel-tensor, curvature, variational principles, etc.) are not at all peculiar to the metrical connexion. They come in in a much simpler, more natural and surveyable fashion when you first only introduce as much of a connexion as, and precisely that kind of connexion for which, the idea of 'differentiation' calls out peremptorily in view of the general invariance you have admitted. That is the so-called *affine* connexion. It is then easy, if desired, to specialize it so as to engender a metric.

An important group of attempts to generalize the 1915 theory (inaugurated by H. Weyl as early as 1918) is based on this more general type of connexion.

We shall therefore investigate the geometry of our continuum in three steps or stages, viz.

(1) when only general invariance is imposed;
(2) when in addition an affine connexion is imposed;
(3) when this is specialized to carry a metric.

And we shall find it useful to keep account of which notions are peculiar to each stage, I mean to say which *are* accessible and meaningful at that stage without our having to go to the next one, but have *no* meaning in the previous one.

Many of the statements and propositions worked out in the following apply to any number n of dimensions. But since we are not dealing with pure mathematics but only intend to show the simplest access to possible geometrical models of space-time, we have at the back of our mind always the case $n = 4$. It would be tedious to repeat again and again: this theorem applies to any number of dimensions. Of more interest and importance is the case when a theorem *is* restricted to $n = 4$; therefore this fact will usually be stressed explicitly.

PART I

THE UNCONNECTED MANIFOLD

CHAPTER I

INVARIANCE; VECTORS AND TENSORS

We envisage a (four-dimensional) continuum whose points are distinguished from each other by allotting a quadruplet of continuous labels x_1, x_2, x_3, x_4 to each of them. However, this first labelling shall have no prerogative over any other one

$$\left.\begin{aligned} x'_1 &= x'_1(x_1 \dots x_4), & x'_2 &= x'_2(x_1 \dots x_4), \\ x'_3 &= x'_3(x_1 \dots x_4), & x'_4 &= x'_4(x_1 \dots x_4), \end{aligned}\right\} \tag{1.1}$$

where the x'_k are four continuous, differentiable functions of the x_k, such that their functional determinant vanishes nowhere.†

But, of course, if such a transformation is made, it must be announced and the functions must be indicated, lest the labelling go to the dogs and the points be 'lost'.

Now we are looking out for mathematical entities, numbers or sets of numbers to which a meaning can be attached in such a manifold.

The numerical values of the coordinates are not of that kind, since they change on transformation, and so would any given mathematical function of them, e.g. the sum of their squares. But on the other hand, if there shall be any meaning in safeguarding the *individuality* of every point even on transformation, we must allow that attached to a point may be some *property* that remains, of course, unchanged on transformation. For unless we intend to enunciate some fact concerning that particular point of space-time, what would be the good of labelling it carefully so as to find it again in any frame? Our list of labels would amount to a list of (grammatical) subjects without predicates; or to writing out an elaborate list of addresses without any intention ever to bother who or what is to be found at these addresses.

In the simplest case such a property will be expressed by one number, attached to the point and, by definition, not changing on

† This is necessary in order to secure a one-to-one correspondence between the two sets of labels. But it is well known that exceptions are quite often put up with, as, for example, in the transition from Cartesian to polar coordinates.

transformation. In the way of *illustration* you may think, for example, of the temperature at a given point of a body at a given time. A property expressed by a number that 'by order' is not to be changed on transformation of the frame is called an *invariant* or a *scalar*. We speak of an invariant *field* or scalar *field*, if not only to one particular point but to every point within a certain region a number is attached, all these numbers referring to the same invariant property. Thus a scalar field will be given by a function of the coordinates

$$\phi(x_1, x_2, x_3, x_4),$$

but not by a definite mathematical function. After the transformation (1.1) the same field will be described by substituting for the x_k their values (functions) obtained from the equations (1.1) by solving them; thus if we call these solutions $x_k(x'_1, x'_2, x'_3, x'_4)$, the field will now, in the new frame, be given by

$$\phi[x_1(x'_1, x'_2, x'_3, x'_4), \quad x_2(x'_1 \dots x'_4), \quad x_3(x'_1 \dots x'_4), \quad x_4(x'_1 \dots x'_4)];$$

and this is, of course, an entirely different function of the x'_k from what ϕ was of the x_k. Strictly speaking, we should indicate it by a different letter, say $\psi(x_1, x_2, x_3, x_4)$. The physicist, however, has taken to regarding a definite letter (ϕ in our case) as referring to a *particular* field in *any* frame. His most important general considerations usually refer to 'the general frame', which he does not specialize and therefore has not actually to change very often, though the principle of invariance on transformation is continually at the back of his mind. Whenever he has to contemplate two or more frames simultaneously, say $x_k, x'_k, x''_k, \dots$, he would choose for the functions describing the same scalar field in these various frames the letters

$$\phi, \phi', \phi'', \dots,$$

so that, for example, in the notation used above,

$$\phi[x_1(x'_1 \dots x'_4), \quad x_2(x'_1 \dots x'_4), \quad x_3(x'_1 \dots x'_4), \quad x_4(x'_1 \dots x'_4)] \equiv \phi'(x'_1, x'_2, x'_3, x'_4).$$

For brevity we shall in future write $\phi(x_k)$ instead of $\phi(x_1, x_2, x_3, x_4)$, if it is at all necessary to indicate the arguments. Usually they can be inferred. Also, the dash in ϕ' would indicate that we mean the field-function expressed in the x'_k-frame, without it being necessary to write $\phi'(x'_k)$.

Given (in one frame) two points, P, with coordinates x_k, and $\bar{P}$, with coordinates $\bar{x}_k$, the difference

$$\phi(\bar{x}_k) - \phi(x_k)$$

is also invariant on transformation. Hence also (taking $\bar{P}$ infinitesimally near to P)

$$\frac{\partial \phi}{\partial x_k} dx_k = \text{invariant.} \tag{1.2}$$

(Throughout these lectures we use the convention that the sum from 1 to 4 is to be understood, whenever the same index appears twice in a product.) Indeed since, on transformation,

$$\frac{\partial \phi}{\partial x'_k} = \frac{\partial \phi}{\partial x_l} \frac{\partial x_l}{\partial x'_k} \tag{1.3}$$

and

$$dx'_k = \frac{\partial x'_k}{\partial x_m} dx_m, \tag{1.4}$$

we get

$$\frac{\partial \phi}{\partial x'_k} dx'_k = \frac{\partial \phi}{\partial x_l} \frac{\partial x_l}{\partial x'_k} \frac{\partial x'_k}{\partial x_m} dx_m = \frac{\partial \phi}{\partial x_l} dx_l.$$

The latter (which proves the statement (1.2)) is obtained by summing over k, since $\frac{\partial x_l}{\partial x'_k} \frac{\partial x'_k}{\partial x_m}$ is the partial derivative of x_l (regarded as a function of the undashed x's) with respect to x_m. And that is 1 or 0 according to whether l is the same index as m or different from it.

The array of the four quantities $\partial\phi/\partial x_k$ is itself a mathematical entity with a definite meaning, provided you subject it to the transformation rule (1.3), just as the scalar ϕ was subject to being *not* transformed but simply 'substituted' (German: *umgerechnet*). The meaning of $\partial\phi/\partial x_k$ is that, in *any* frame, it gives you the increment of ϕ (on proceeding to a neighbouring point) as the sum of products, indicated in (1.2), the increments of the coordinates to be taken, of course, in that frame. The entity described by these four partial derivatives is called the gradient of ϕ and is the first example of a property referring to a definite point and given not by one number only, as a scalar is, but by an array of numbers, four in this case. It is the prototype of a *covariant vector*. More especially, it is a covariant vector *field*.

The general conception of a covariant vector is an array of four quantities A_k which 'by order' is to be transformed according to (1.3), thus:

$$A'_k = \frac{\partial x_l}{\partial x'_k} A_l. \tag{1.5}$$

The nature of the entity may (as in the case of the gradient) be such that there is a quadruplet of numbers attached to every point, varying from point to point. Then we speak of a *field*. Or the particular vector might refer just to *one* point. But at all events every vector must refer to *one definite* point, otherwise the prescription (1.5) would be meaningless, we should not know what coefficients to use in it. (What has just been said will refer in the same way and for the same reasons to all the other vectors and tensors to be introduced presently.)

The way in which, according to (1.4) the *differentials of the coordinates* transform is a sort of counterpart to (1.3). We define a *contravariant vector* as an array of four quantities B^k which transform in the same fashion as the dx_k:

$$B'^k = \frac{\partial x'_k}{\partial x_m} B^m. \tag{1.6}$$

By general convention the writing of the index as subscript or as superscript respectively serves to distinguish the 'covariant' and 'contravariant' behaviour. The dx_k themselves are thus an (infinitesimal) contravariant vector, indeed its prototype. With regard to our convention some people write x^k instead of x_k for the coordinates. I do not think this makes for consistency since (i) the x_k themselves are no vector at all and (ii) the symbols $\partial/\partial x_k$ can in many respects be regarded as a (symbolic) *co*variant entity. So it is better to remember that in all these cases the position of the *whole differential* (whether it stands in the numerator or in the denominator) replaces, as it were, the position of the *index*.

From (1.5) and (1.6) follows immediately

$$A'_k B'^k = A_k B^k = \text{invariant}. \tag{1.7}$$

It is called the inner or scalar product. When it is zero, the two vectors are by some people called pseudo-orthogonal.

Given several (s) vectors at the same point, partly covariant, partly contravariant, the array of 4^s quantities

$$A^k B^l C^m \dots G_p H_q \dots \tag{1.8}$$

follows a linear transformation law that can easily be made out from (1.5) and (1.6), but we need not write it out explicitly. An array of 4^s quantities which follow *this* transformation law is called a tensor of rank s and indicated by a symbol like

$$T^{klm\dots}{}_{pq\dots}, \tag{1.9}$$

where, of course, the number of superscripts and the number of subscripts must be given separately, fully to characterize the nature of the entity T. The product (1.8) is a special case of such a tensor, but not the most general tensor of this kind, since it depends only on $4s$ independent numbers and

$$4s < 4^s$$

for $s > 1$. The order of the superscripts in the notation (1.9) is relevant, indeed $T^{lkm\dots}{}_{pq\dots}$ would in the particular case (1.8) mean $A^l B^k C^m \dots G_p H_q \dots$, which is different from (1.8).

It is not the same tensor, but it is a tensor of the same type. It is worth showing that it really has *exactly* the same transformation rule. An example will suffice. Take a contravariant tensor of the third rank T^{klm}. It transforms thus:

$$T'^{klm} = \frac{\partial x'_k}{\partial x_r}\frac{\partial x'_l}{\partial x_s}\frac{\partial x'_m}{\partial x_t} T^{rst}.$$

Exchange k and l and at the same time the notation of the summation indices r, s

$$T'^{lkm} = \frac{\partial x'_l}{\partial x_s}\frac{\partial x'_k}{\partial x_r}\frac{\partial x'_m}{\partial x_t} T^{srt}.$$

The coefficient is unchanged, but in the T-arrays the first two indices have been exchanged. The point is that you may regard the T^{123} component as the (213)-component etc. of *another* tensor. The same would hold for any permutation, provided you make the same permutation in *all* components.

The same holds, of course, for subscripts. But at the moment there is no relevant order between subscripts and superscripts.

The two types of *vectors* are clearly special cases of tensors, viz. the tensors of rank 1. A scalar may be called a tensor of rank zero.

By multiplying the components of any two tensors in all combinations:

$$T^{klm\dots}{}_{pq\dots} S^{abc\dots}{}_{rst\dots}$$

you get again a tensor. That is clear from the transformation rules. It is called the outer or direct product of the two.

If in (1.9) you execute a summation with respect to an upper and a lower index, as for example,

$$T^{klm\ldots}{}_{kq\ldots}, \tag{1.10}$$

it is again easy to show from the transformation rule (which we have indicated, but not written out) that this is a tensor with rank two less than the original one. It could be indicated by a symbol like

$$S^{lm\ldots}{}_{q\ldots}. \tag{1.11}$$

This process of forming from a given tensor which has at least one index of each kind a tensor of lower rank is called contraction (German: *Verjüngung*). Observe that (1.9) admits of various contractions. The tensor, e.g.

$$T^{klm\ldots}{}_{pk\ldots}, \tag{1.12}$$

is distinctively different from (1.10), though of the same general type, i.e. the same rank and the same number of superscripts and subscripts.

Tensors can be added or subtracted or, more generally, linearly combined with either constant or invariant (scalar) coefficients, if and only if they are of exactly the same type and refer to exactly the same point of the continuum. By 'can be' we mean that in this and only in this case, the result will again have a simple transformation formula, to wit it is a tensor of the same type and referring to the same point.

The most important number in mathematics is the zero. Our present sign for it as well as the word zero comes from the Arabs. (It is, by the way etymologically the same as English cipher, French *chiffre*, German *Ziffer*, which have, however, acquired a different meaning.) But the notion is older, it turns up in Babylonian Mathematics soon after 1000 B.C.† and may have been received from India. Let me dwell for a moment on the importance of this concept. A great many of our propositions and statements in mathematics take the form of an equation. The essential enunciation of an equation is always this: that a certain number is zero. Zero is the only number with a charter, a sort of royal privilege.

† V. Gordon Childe, *Man Makes Himself* (London: Watts and Co., 1936), pp. 222 and 255.

While with any other number any of the elementary operations may be executed, it is prohibited to *divide* by zero—just as, for example, in many houses of parliament *any* subject may be discussed, only the person of the sovereign is excluded. If you divide by zero, nonsense is usually the result. This prerogative is essential, you have to think of it every minute; whenever you divide, you must satisfy yourself that the divisor is not 'of royal blood', that it is *not* zero. Another consequence is that royal blood cannot (by multiplication) be obtained otherwise than from royal blood. A product cannot vanish unless at least one of its factors vanish. It is not accidental that more often than not the conclusion of a proof runs thus: $AB = 0,\ B \neq 0,\ \therefore\ A = 0$.

In the same way the most important tensor of any type is the zero tensor of that type, that is, the one whose components all vanish. It is a *numerically invariant tensor*, since the transformation formulae are linear and homogeneous. That is the reason why tensors play the all-important part they play. For it has the consequence that an equation of the following kind between two tensors S and T

$$S^{kl\ldots}{}_{pq\ldots} = T^{kl\ldots}{}_{pq\ldots}$$

is independent of the frame (for it means that $S^{\cdots}{}_{\cdots} - T^{\cdots}{}_{\cdots}$ is the zero tensor)—provided, of course, that S and T are of the same type and refer to the same point. If they did not, this would not hold, the above equation would be meaningless, and therefore we shall never contemplate that sort of thing.

Perhaps this is the place to mention a convention, which is always made *tacitly*, though it would deserve to be mentioned explicitly, just as the 'summation convention', of which it is the counterpart. According to the latter an index that appears twice in the same product is to include summation from 1 to 4. Now an index that appears only once, but then, of course, in *every term* of an equation, implies that the equation holds for any value 1 to 4 of that index. By the first convention we shove many terms of an equation into *one*, by the second we shove many equations into one. For example, if you write

$$S^{klm}{}_{m} = R^{kl},$$

this represents in general 16 equations, each of which has four terms on the left.