CHAPTER 1. FUNCTIONS OF ONE COMPLEX VARIABLE

Introduction

Our aim in this chapter is to develop the familiar theories of analytic functions of one complex variable and Riemann surfaces in a way that generalises well to the several variable theory. In §3 we see how the existence theory of the Cauchy-Riemann equations can be used to prove the Mittag-Leffler theorem and also how the topology of domains in $\mathbb{C}$ naturally enters into the proof of the Weierstrass theorem. In §§4,5 we show how the theory of holomorphic line bundles may be used to reformulate some of the classical problems in Riemann surface theory. We also define the Cauchy-Riemann equations on an arbitrary Riemann surface and indicate how they are related to the problem of constructing meromorphic functions with specified divisors. In an appendix we prove a number of classical results, including the Runge approximation theorem. We use the Runge theorem to construct solutions of the Cauchy-Riemann equations.

§1. Analytic functions and power series

Let Ω be a domain in $\mathbb{C}$. We recall that a function $f: \Omega \to \mathbb{C}$ is said to be *analytic* or *holomorphic* if it is complex differentiable on Ω. Writing f in real and imaginary parts, $f = u + iv$, analyticity implies that u and v satisfy the Cauchy-Riemann equations on Ω:

$$\partial u/\partial x = \partial v/\partial y; \quad \partial u/\partial y = -\partial v/\partial x .$$

Recalling that a real 2×2-matrix $[a_{ij}]$ induces a complex linear endomorphism of $\mathbb{C}$ if and only if $a_{11} = a_{22}$ and $a_{12} = -a_{21}$, we may interpret the Cauchy-Riemann equations as saying that if f is analytic then f is differentiable in the real sense and the (real) derivative of f is everywhere a complex linear map (see, for example, Field [1; Example 3, page 133]).

Let $A(\Omega)$ denote the set of all analytic functions on Ω.

We now introduce a pair of partial differential operators which, together with their generalisations to several variables, will be of the utmost importance in the sequel. We set

$$\partial/\partial z = \tfrac{1}{2}(\partial/x - i\partial/\partial y); \quad \partial/\partial\bar{z} = \tfrac{1}{2}(\partial/\partial x + i\partial/\partial y).$$

For $r \geq 1$,

$$\partial/\partial z, \partial/\partial\bar{z}: C^r(\Omega) \to C^{r-1}(\Omega).$$

The significance of these operators may be gauged from

Lemma 1.1.1. A function $f \in C^1(\Omega)$ is analytic if and only if $\partial f/\partial\bar{z} = 0$.

Proof. The reader may verify that $\partial f/\partial\bar{z} = 0$ iff the Cauchy-Riemann equations hold. Since f is assumed to be C^1, the Cauchy-Riemann equations hold iff f is analytic. □

Next we recall the basic theorem on the local representation of analytic functions by power series.

Theorem 1.1.2. Let $f \in A(\Omega)$. Given $\zeta \in \Omega$ and $r > 0$ such that $D_r(\zeta) \subset \Omega$, we have

$$f(z) = \sum_{j=0}^{\infty} a_j (z - \zeta)^j, \quad z \in D_r(\zeta),$$

where $a_j = \partial^j f/\partial z^j(\zeta)/j!$, and convergence is uniform on compact subsets of $D_r(\zeta)$.

Remark. Simple examples, such as $f(z) = (1-z)^{-1}$, $\Omega = \mathbb{C} \setminus \{1\}$, show that the power series at ζ need not converge on the whole of Ω.

Corollary 1.1.3. An analytic function is C^∞.

Remark. Notice that $A(\Omega) = \mathrm{Kernel}(\partial/\partial\bar{z})$. Now $\partial/\partial\bar{z}$ is an example of an *elliptic* differential operator and it can be shown that the kernel of any elliptic operator consists of C^∞ functions. We shall return to this type of question in later chapters.

Corollary 1.1.4. If $f \in A(\Omega)$, then $\partial^j f/\partial z^j \in A(\Omega)$, $j \geq 0$.

Corollary 1.1.5. Let Ω be a domain in $\mathbb{C}$. Suppose $f \in A(\Omega)$ and that at some point $\zeta \in \Omega$, $\partial^j f/\partial z^j(\zeta) = 0$, $j \geq 0$. Then f vanishes identically on Ω.

Proof. Let $X = \{z \in \Omega: \partial^j f/\partial z^j = 0, \text{ all } j \geq 0\}$. X is open by the power series representation of analytic functions given by Theorem 1.1.2 and X is certainly non-empty since $\zeta \in X$. Since X is the intersection of the closed sets $\{z \in \Omega: \partial^j f/\partial z^j(z) = 0\}$, X is also closed. Since Ω is connected, $X = \Omega$. □

Remark. Another way of stating this corollary is that the value of an analytic function and all its derivatives at a single point of a domain determine the function uniquely. This type of behaviour does not, of course, hold for C^∞ or continuous functions.

Proposition 1.1.6. (Uniqueness of analytic continuation). Let U, V be connected open subsets of $\mathbb{C}$ and suppose $U \cap V \neq \emptyset$. If $f \in A(U)$ and h is an analytic extension of f to $U \cup V$ (that is, $h \in A(U \cup V)$ and $h|U = f$), then h is unique.

Proof. If h_1, h_2 are analytic extensions of f to $U \cup V$ then $h_1 - h_2$ is an analytic extension of the zero function on U to $U \cup V$. Therefore, $h_1 - h_2$ is identically zero by Corollary 1.1.5. □

Remark. Once we have uniqueness of analytic continuation it is natural to try to construct the largest domain to which any given analytic function may be extended. It turns out of course that we have to enlarge our class of domains to include Riemann surfaces spread over $\mathbb{C}$. We return to this question in §4 of this chapter and again in §2 of Chapter 6.

Exercises. These exercises are revision of basic theory of functions of one complex variable. Proofs may be found in any of the many introductory texts on complex analysis.

1) (Laurent series). Let f be analytic on the annulus $r < |z - z_0| < R$. Derive the Laurent series of f at z_0,

$$f(z) = \sum_{n=-\infty}^{n=+\infty} a_n(z - z_0)^n, \quad r < |z - z_0| < R,$$

where $a_n = (2\pi i)^{-1} \int_{|\zeta - z_0| = s} f(\zeta)/(\zeta - z_0)^{n+1} d\zeta$, $r < s < R$, and convergence is uniform on compact subsets of the annulus.

2) (Cauchy's inequalities). Continuing with the notation and assumptions of question 1, show that if $M(t) = \sup\{|f(z)|: |z - z_0| = t\}$, $r < t < R$, then

$$|a_n| \leq M(t)/t^n, \ n \in \mathbb{Z} \ .$$

In particular, show that if f is holomorphic on the disc $D_R(z_0)$ then

$$|\partial^n f/\partial z^n(z_0)| \leq M(t)n!/t^n, \ n \geq 0 \ .$$

3) (Riemann removable singularities theorem). Suppose that f is an analytic function in the punctured disc $D_r(z_0)^* = \{z: 0 < |z - z_0| < r\}$. Show that a necessary and sufficient condition for f to extend analytically to $D_r(z_0)$ is that f is bounded on some neighbourhood of z_0. (Use the result of question 2).

4) (Open mapping theorem). Let f be analytic and not identically zero on the domain U in $\mathbb{C}$. Prove that f(U) is open in $\mathbb{C}$.

5) (Monodromy theorem). Let Ω be a domain in $\mathbb{C}$, $z_0, y_0 \in \Omega$ and suppose f is analytic on some neighbourhood U of z_0. Let C be a continuous path in Ω parametrized by $\phi: [0,1] \to \Omega$ with $\phi(0) = z_0$, $\phi(1) = y_0$. We say f can be analytically continued along C if we can find discs $D_i = D_{r_i}(\phi(t_i))$, $0 = t_0 < \ldots < t_k = 1$, covering C, $h_i \in A(D_i)$, $i = 0,\ldots,k$, such that $h_0 = f$ on $U \cap D_0$ and $h_i = h_{i+1}$ on $D_i \cap D_{i+1}$, $i \geq 0$. Define $f_C(y_0)$ to be $h_k(y_0)$. Show that

a) $f_C(y_0)$ depends only on C and not on any of the choices we have made.

b) If C, C' are homotopic curves in Ω joining z_0 to y_0 then $f_C(y_0) = f_{C'}(y_0)$.

Give examples to show that if Ω is not simply connected and C, C' are non-homotopic curves joining z_0 to y_0 then $f_C(y_0)$ may not equal $f_{C'}(y_0)$.

6) (Maximum principle). Let f be analytic on the domain Ω in $\mathbb{C}$. If $|f|$ has a maximum in Ω then f is constant.

7) (Schwarz' lemma). Suppose $f \in A(D_1(0))$ and satisfies $f(0) = 0$ and $|f(z)| \le 1$, $z \in D_1(0)$. Then $|f(z)| \le z$ and $|f'(0)| \le 1$. Equality holds if and only if $f(z) = cz$, where $|c| = 1$ (Hint: Apply the maximum principle to the function $f(z)/z$).

§2. Meromorphic functions

Roughly speaking meromorphic functions are the analytic analogue of rational functions and in this section we briefly review their definition and elementary properties. Throughout the section Ω will denote a domain in $\mathbb{C}$.

Let $\zeta \in \Omega$ and $\Omega' = \Omega \setminus \{\zeta\}$. Suppose $f \in A(\Omega')$. By Laurent's expansion we have

$$f(z) = \sum_{j=-\infty}^{j=+\infty} a_j (z-\zeta)^j, \quad z \in D_r(\zeta)^* \subset \Omega.$$

There are three possibilities:

a) f is bounded on some neighbourhood of ζ. In this case $a_j = 0$, $j < 0$, and f extends uniquely to an analytic function on Ω (Riemann removable singularities theorem).

b) $f(z) \to \infty$ as $z \to \zeta$. Here one can show that there exists a strictly positive integer N such that $a_j = 0$, $j < -N$ and $a_{-N} \neq 0$.

c) Neither a) or b) occurs (ζ is an *essential singularity* of f).

In case b), f is an example of a *meromorphic function* on Ω with a single *pole* of order N at ζ. We may write $f(z) = u(z)/(z-\zeta)^N$, $z \in D_r(\zeta)^*$, where $u \in A(D_r(\zeta)^*)$ and the power series of u at ζ is given explicitly as the Laurent series of f at ζ multiplied by $(z-\zeta)^N$. We may extend u analytically to Ω by taking $u(z) = (z-\zeta)^N f(z)$, $z \neq \zeta$. In this way we may represent f as the quotient $u(z)/(z-\zeta)^N$ of analytic functions defined on all of Ω.

There are some problems in giving a satisfactory general definition of a meromorphic function. If we attempt to define a meromorphic function on Ω as a quotient f/g, $f,g \in A(\Omega)$, $g \not\equiv 0$, we are faced with the difficulty that f and g may have infinitely many common zeros. If this happens we canot cancel the zeros using the elementary power series techniques of the previous paragraph to obtain the maximal subdomain of Ω on which the meromorphic function is defined as an analytic function. That is, the representation f/g may be rather non-canonical. We start by giving a definition that is rather special to functions of one complex variable and then show how to reformulate the definition in a way that generalises well to functions of several complex variables.

Definition 1.2.1. We say that m is a *meromorphic* function on Ω if there exists a discrete subset X of Ω such that

i) m is an analytic function on $\Omega \setminus X$.

ii) Every point of X is a pole of m.

We notice that the definition excludes essential singularities and so, for example, $e^{-1/z}$ would not define a meromorphic function on $\mathbb{C}$.

We denote the set of meromorphic functions on Ω by $M(\Omega)$.

Locally a meromorphic function can be expressed as a quotient of analytic functions. That is, given $m \in M(\Omega)$ and $z \in \Omega$, we can find an open neighbourhood U of z and $f,g \in A(U)$ such that g is not identically zero and $m|U = f/g$ outside of any poles of m in U. Of course, if U does not contain any poles of m, we can take $g \equiv 1$.

We now work towards an alternative definition of a meromorphic function which is framed in terms of local information and requires no explicit information about the pole set.

Suppose that $\{U_i : i \in I\}$ is an open cover of Ω and that for each $i \in I$ we are given $f_i, g_i \in A(U_i)$ with g_i not vanishing identically on any connected component of U_i. Then $\{(f_i, g_i): i \in I\}$ defines a meromorphic function m on Ω provided that for all $i,j \in I$ we have

$$f_i/g_i = f_j/g_j$$

at all points of $U_i \cap U_j$ where both g_i and g_j are non-zero. (Equivalently, $f_i g_j = f_j g_i$ on $U_i \cap U_j$). We omit the routine construction of m. If

$\{V_j: j \in J\}$ is another open cover of Ω and $\{(a_j,b_j): j \in J\}$ a corresponding set of analytic functions satisfying the above conditions, it is easily verified that $\{(f_i,g_i): i \in I\}$ and $\{(a_j,b_j): j \in J\}$ define the same meromorphic function if and only if

$$f_i/g_i = a_j/b_j$$

at all points of $U_i \cap V_j$ where both g_i and b_j are non-zero.

We can now use this condition to define an equivalence relation on all sets of pairs of analytic functions $\{(f_i,g_i): i \in I\}$ satisfying the requisite compatibility conditions. The equivalence classes of this relation are then defined to be meromorphic functions. This is essentially the approach that we adopt in later chapters.

One immediate consequence of our local description of meromorphic functions is that $M(\Omega)$ is a field (the connectedness of Ω is essential here to avoid zero divisors).

Suppose $m \in M(\Omega)$ and $\zeta \in \Omega$. We define the order of m at ζ, $\mathrm{ord}(m,\zeta)$, to be the smallest index with non-zero coefficient in the Laurent expansion of m at ζ. That is, if

$$m(z) = \sum_{j=N}^{\infty} a_j(z-\zeta)^j$$

on some neighbourhood of ζ and $a_N \neq 0$, then $\mathrm{ord}(m,\zeta) = N$. Clearly, if $m = f/g$ in some neighbourhood of ζ, $\mathrm{ord}(m,\zeta) = \mathrm{ord}(f,\zeta) - \mathrm{ord}(g,\zeta)$ though of course the terms on the right hand side depend on the choice of local representation for m!

If $\mathrm{ord}(m,\zeta) > 0$, we say that m has a *zero of order* $\mathrm{ord}(m,\zeta)$ at ζ and if $\mathrm{ord}(m,\zeta) < 0$, we say that m has a *pole of order* $-\mathrm{ord}(m,\zeta)$ at ζ. We set

$$Z(m) = \{z \in \Omega: \mathrm{ord}(m,z) > 0\} = \{z \in \Omega: m(z) = 0\}$$

$$P(m) = \{z \in \Omega: \mathrm{ord}(m,z) < 0\} = \{z \in \Omega: m \text{ has a pole at } z\} .$$

$Z(m)$ and $P(m)$ are called the zero and pole set of m respectively. Clearly $Z(m)$ and $P(m)$ are disjoint discrete subsets of Ω (assuming $m \neq 0$).

We now introduce some useful definitions and notation. Suppose $p: \Omega \to \mathbb{Z}$ and $\{z \in \Omega: p(z) \neq 0\}$ is a discrete subset of Ω. We call the formal sum $\sum_{z\in\Omega} p(z).z$ a *divisor* on Ω. We denote the set of divisors on Ω by $\mathcal{D}(\Omega)$. $\mathcal{D}(\Omega)$ has the structure of an ordered abelian group if we define

$$\left(\sum_{z\in\Omega} p(z).z \pm \sum_{z\in\Omega} q(z).z\right) = \sum_{z\in\Omega} (p \pm q)(z).z$$

$$\sum_{z\in\Omega} p(z).z > \sum_{z\in\Omega} q(z).z \quad \text{if and only if}$$

$p(z) \geq q(z)$ for all $z \in \Omega$ with strict inequality for at least one point of Ω.

Let $M^*(\Omega)$ denote the group of invertible elements of $M(\Omega)$. Since Ω is assumed connected, $M^*(\Omega)$ is all of $M(\Omega)$ except the zero function. Given $m \in M^*(\Omega)$, the divisor of m, $\mathrm{div}(m)$, is defined to be

$$\sum_{z\in\Omega} \mathrm{ord}(m,z).z$$

$\mathrm{div}: M^*(\Omega) \to \mathcal{D}(\Omega)$ is a homomorphism (relative to the multiplicative structure on $M^*(\Omega)$). We note that $\mathrm{div}(m) \geq 0$ if and only if $m \in A(\Omega)$.

Suppose $m \in M(\Omega)$ and $\zeta \in \Omega$. Let m have Laurent series

$$\sum_{j=N}^{\infty} a_j(z-\zeta)^j$$

at ζ, where we suppose $a_N \neq 0$. The principal part of m at ζ is defined to be

$$\sum_{j=N}^{-1} a_j(z-\zeta)^j$$

if $N \leq -1$ and zero otherwise. We note that $m,m' \in M(\Omega)$ have the same principal part at ζ if and only if $m - m'$ is analytic on some neighbourhood of ζ.

To conclude this section we remark that Proposition 1.1.6 generalises to meromorphic functions and therefore we can discuss meromorphic continuation. We leave details to the reader.

Exercises

1. Verify that

$$\mathrm{div}(mm') = \mathrm{div}(m) + \mathrm{div}(m'),\ m,m' \in M^*(\Omega).$$

$$\mathrm{div}(m^{-1}) = -\mathrm{div}(m),\ m \in M^*(\Omega).$$

$\mathrm{div}(m) = 0$ if and only if m is a nowhere vanishing analytic function on Ω.

2. Let $m \in M^*(\mathbb{C})$. Show that if $\mathrm{div}(m) = 0$ then either m is constant or m has an essential singularity at infinity.

3. Under what conditions is the composition of two meromorphic functions metomorphic?

§3. Theorems of Weierstrass and Mittag-Leffler

In the preceding sections we have reviewed some of the basic elementary properties of analytic and meromorphic functions. However, we have as yet given no means of constructing such functions so as to satisfy specified properties. For example, if X is any closed subset of $\mathbb{C}$ it is not difficult to construct a C^∞ function on $\mathbb{C}$ with zero set X. Can we find an analytic function whose zero set is equal to X? Clearly we cannot unless X is a discrete subset of $\mathbb{C}$. It turns out though that if X is discrete we can always find an analytic function on $\mathbb{C}$ with zero set X. This is exactly the type of result we need if our study of analytic functions is to amount to much more than a study of polynomials, rational functions and the standard analytic functions such as log and exp.

Our aim in this section will be to show the importance of the theory of the partial differential operator $\partial/\partial\bar{z}$ and the topology of domains in $\mathbb{C}$ in questions involving the construction of analytic and meromorphic functions with specified behaviour at prescribed poles and zeros. We adopt this approach because it generalises well to functions of several complex variables and complex manifolds. We must emphasise, however, that the Mittag-Leffler and Weierstrass theorems can be given rather more elementary proofs than those presented here which do not depend on the theory of $\partial/\partial\bar{z}$ (see, for example, Heins [1] or Hille [1]).

Throughout this section Ω will denote an open subset of $\mathbb{C}$.

We give the proof of the following basic existence theorem in the appendix to this chapter.

Theorem 1.3.1. Let $f \in C^\infty(\Omega)$. Then there exists $u \in C^\infty(\Omega)$ such that

$$\partial u/\partial\bar{z} = f .$$

Remark. An equivalent formulation of Theorem 1.3.1 is that the sequence

$$0 \to A(\Omega) \xrightarrow{i} C^\infty(\Omega) \xrightarrow{\partial/\partial\bar{z}} C^\infty(\Omega) \to 0$$

is exact for every open subset Ω of $\mathbb{C}$ (i denotes inclusion).

The Mittag-Leffler theorem gives conditions under which there exists a meromorphic function on Ω with specified principal parts.

Before stating the Mittag-Leffler theorem we introduce some useful notation. Suppose U_i, U_j are sets, then we let U_{ij} denote the intersection $U_i \cap U_j$. We use the obvious generalisation of this notation for intersections of more than two indexed sets.

Theorem 1.3.2. (Mittag-Leffler theorem) Let $\{U_i: i \in I\}$ be an open cover of Ω and suppose we are given $m_i \in M(U_i)$ for each $i \in I$. Then, provided that $m_i - m_j \in A(U_{ij})$ for all $i,j \in I$, there exists $m \in M(\Omega)$ such that

$$m - m_i \in A(U_i),\ i \in I .$$

Remark. An alternative formulation of this theorem would be: Suppose X is a discrete subset of Ω and that for each $z \in X$ we are given a meromorphic function m^z which is defined on some neighbourhood of z and has a single pole at z. Then there exists $m \in M(\Omega)$ with pole set X and such that the principal part of m at z equals the principal part of m^z at z for all $z \in X$.