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# Numerical Ranges II

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# Introduction

This volume is a sequel to Numerical ranges of operators on normed spaces and of elements of normed algebras, which is here denoted by NRI. Although it appeared in 1971, NRI was written in 1969, and since then the subject has made vigorous progress, reaching a high point withthe Conference on Numerical Ranges, held in Aberdeen, July 1971. This conference gave us an unusually good opportunity to see the scope of the subject, which is much less specialized than the title might suggest.

A comparison of the present volume with NRI will show that the theory of numerical ranges has become immensely richer both in depth and in width. The contents have been grouped into three chapters: 5. Spatial numerical ranges; 6. Algebra numerical ranges; 7. Further ranges.

In Chapter 5 we are mainly concerned with the improvement of the Bishop-Phelps theorem due to Bollobás [115] and with applications of this useful tool. We also give the remarkable theorem of Zenger [78] on the inclusion of the convex hull of the point spectrum in the spatial numerical range V(T), and the equally remarkable results of Crabb [136] and Sinclair [202] concerning points of $\mathrm{Sp}(T) \cap \partial V(T)$.

NRI contained an inequality (Theorem 4. 8) relating the norms of iterates to the numerical radius and a remark that this inequality had been proved to be best possible in a strong sense. Chapter 6 contains a systematic approach to such best possible inequalities through the theory of the extremal algebra Ea(K) which has been developed by Bollobás [117] and Crabb, Duncan, and McGregor [134]. Here the numerical range comes into contact with interesting function theoretic ideas. Chapter 6 also contains an account of the striking progress made in the study of Hermitian elements and related concepts by Berkson [109], Browder [125], Berkson, Dowson and Elliott [113], Moore [186], Sinclair [204], and others. The proof of the Vidav-Palmer theorem, which bulked large in NRI, receives some final touches.

In our final chapter we give a brief survey of essential numerical ranges, joint numerical ranges, and matrix ranges, and end with an axiomatic approach to numerical ranges. The theory of the essential numerical range has been developed with force by Fillmore, Stampfli and Williams [151], and by Anderson [100] who has established the important operator theoretic significance of the condition $0 \in \text{Wess}(T)$. Two concepts of matrix range have been developed, the analogue of the algebra numerical range by Arveson [104], and the analogue of the spatial numerical range by S. K. Parrott (unpublished), and both concepts have been shown to provide complete sets of unitary invariants for certain wide classes of compact operators.

We are very much aware that our account of numerical ranges remains unbalanced in that we have not attempted to give an account of the applications to initial value problems. When we came to study the literature of this important subject, we soon concluded that we were not qualified to do it justice, and we hope that some expert in the field will fill this gap. A valuable bibliography is given in Calvert and Gustafson [129]. We have also refrained from developing numerical ranges in real algebras; significant advances in this area may be found in Lumer [176] and McGregor [180]. The bibliography in NRI lacked an adequate coverage of numerical ranges for Hilbert space operators and we have attempted to repair this deficiency in the present volume. The present bibliography also contains several other items which are not mentioned in the text.

This volume being a companion to NRI, we have continued the same mode of references. To simplify back references we have numbered the sections in NRII starting with §15. Likewise the bibliography in NRII starts with [100], so that [n] with $n < 100$ refers to the bibliography in NRI.

Many authors have given us valuable help by making their work available to us before publication, and we wish to acknowledge particularly the help received from W. B. Arveson, E. Berkson, B. Bollobás, A. Browder, M. J. Crabb, H. R. Dowson, K. Gustafson, L. A. Harris, C. M. McGregor, R. T. Moore, T. W. Palmer, S. K. Parrott, A. M. Sinclair, and J. P. Williams. As in NRI only a very few results appear in print here for the first time. Most of the material in this volume has been the subject of seminar talks by the authors, and has benefitted from the resulting criticism. We have had many valuable conversations with G. A. Johnson which have left their mark particularly on §§36, 37. In the elimination of errors we have been greatly helped by D. J. Baker and A. W. Tullo who have read the manuscript.

The whole manuscript has been most expertly typed by Miss Christine Bourke.

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