5 · Spatial numerical ranges

15. SOME ELEMENTARY OBSERVATIONS ON THE SPATIAL NUMERICAL RANGE

Let X denote a normed linear space over $\mathbb{C}$, X' its dual space, $S(X)$ its unit sphere, $\Pi(X)$ the subset of $X \times X'$ defined by

$$\Pi(X) = \{(x, f) \in S(X) \times S(X') : f(x) = 1\}.$$

Given $x \in S(X)$, let $D(X, x) = \{f \in S(X') : f(x) = 1\}$.

We recall that the spatial numerical range $V(T)$, for $T \in B(X)$, is defined by $V(T) = \{f(Tx) : (x, f) \in \Pi(X)\}$. Given $x \in S(X)$, let

$$V(T, x) = \{f(Tx) : f \in D(X, x)\}.$$

Clearly $V(T) = \bigcup\{V(T, x) : x \in S(X)\}$.

We collect a few elementary observations on $V(T)$, $V(T, x)$ and $\Pi(X)$ into this section. The first is derived from a remark of J. P. Williams on the algebra numerical range.

Lemma 1. *Let* $x \in S(X)$, $T \in B(X)$. *Then* $V(T, x)$ *is the set of all complex numbers* λ *such that*

$$|\lambda - \zeta| \le \|(T - \zeta I)x\| \qquad (\zeta \in \mathbb{C}). \tag{1}$$

Proof. If $\lambda \in V(T, x)$, there exists $f \in D(X, x)$ with $f(Tx) = \lambda$, and so

$$|\lambda - \zeta| = |f((T - \zeta I)x)| \le \|(T - \zeta I)x\| \qquad (\zeta \in \mathbb{C}).$$

Suppose on the other hand that λ satisfies the inequalities (1). If $Tx \in \mathbb{C}x$, we take $\zeta \in \mathbb{C}$ with $Tx = \zeta x$, i.e. with $(T - \zeta I)x = 0$. Then (1) gives $\lambda = \zeta = f(Tx)$ for arbitrary $f \in D(X, x)$. Suppose then that

x, Tx are linearly independent and define f_0 on their linear span by

$$f_0(\alpha Tx + \beta x) = \alpha\lambda + \beta \qquad (\alpha, \beta \in \mathbb{C}).$$

The inequalities (1) imply that $\|f_0\| \le 1$, and we have $f_0(x) = 1$, $f_0(Tx)=\lambda$. The Hahn-Banach theorem now gives an extension f of f_0 with $f \in D(X, x)$ and $f(Tx) = \lambda$.

Remark. This lemma exhibits $V(T, x)$ as an intersection of closed discs.

Lemma 2. *Let X_1 be a non-zero linear subspace of X, let $T \in B(X)$, and let $TX_1 \subset X_1$. Then*

(i) $V(T|_{X_1}) \subset V(T)$,

(ii) $V(T|_{X_1}, x_1) = V(T, x_1) \qquad (x_1 \in S(X_1))$.

Proof. (i) follows from (ii) which is an immediate corollary of Lemma 1.

Remarks. (1) If X_1 is a closed linear subspace of X and $X_1 \ne X$, then the difference space $Y = X - X_1$ is a non-zero normed space with respect to the canonical norm $\|y\| = \inf\{\|x\| : x \in y\}$ $(y \in Y)$. Given $T \in B(X)$ with $TX_1 \subset X_1$, we obtain an operator $U \in B(Y)$ given by $Uy = Tx + X_1$ $(x \in y \in Y)$, and it is natural to ask whether $V(U) \subset V(T)$. Using rather deeper arguments, we prove in Theorem 17.5 that if X is complete then $V(U) \subset V(T)^-$.

(2) Let X_1, X_2 be non-zero subspaces of X such that $X = X_1 \oplus X_2$, and suppose that

$$\|x_1 + x_2\| = \|x_1\| + \|x_2\| \qquad (x_1 \in X_1,\ x_2 \in X_2).$$

Let $T \in B(X)$ with $TX_1 \subset X_1$, $TX_2 \subset X_2$. Then

$$V(T) = \{\alpha\lambda + (1-\alpha)\mu : \alpha \in [0,1],\ \lambda \in V(T|_{X_1}),\ \mu \in V(T|_{X_2})\}.$$

The same conclusion holds if, for some p with $1 \le p \le \infty$, we have

$$\|x_1 + x_2\| = (\|x_1\|^p + \|x_2\|^p)^{1/p} \qquad (x_1 \in X_1,\ x_2 \in X_2).$$

Lemma 3. Let X_R denote the space X regarded as a normed linear space over $\underset{\sim}{R}$. Then the mapping $f \to \mathrm{Re}\, f$ is an isometric real linear mapping of X' onto X_R'.

Proof. Given $f \in X'$, it is clear that $\mathrm{Re}\, f \in X_R'$ and that $\|\mathrm{Re}\, f\| \le \|f\|$. Given $\zeta \in \underset{\sim}{C}$ with $|\zeta| = 1$, we have

$$|\mathrm{Re}(\zeta f(x))| = |\mathrm{Re}\, f(\zeta x)| \le \|\mathrm{Re}\, f\|\ \|\zeta x\| = \|\mathrm{Re}\, f\| \,.\, \|x\|.$$

Since we may choose such a complex number ζ with $\mathrm{Re}(\zeta f(x)) = |f(x)|$, this gives $|f(x)| \le \|\mathrm{Re}\, f\| \,.\, \|x\|$, and so $\|f\| = \|\mathrm{Re}\, f\|$.

It is clear that the mapping $f \to \mathrm{Re}\, f$ is a real linear mapping, and so it only remains to prove that the mapping is surjective. Given $g \in X_R'$, define f by

$$f(x) = g(x) - ig(ix) \qquad (x \in X).$$

Then $f \in X'$, and $\mathrm{Re}\, f = g$.

Corollary 4. The mapping $(x, f) \to (x, \mathrm{Re}\, f)$ maps $\Pi(X)$ onto $\Pi(X_R)$.

Proof. Given $(x, f) \in \Pi(X)$, we have $\|\mathrm{Re}\, f\| = 1$ and $(\mathrm{Re}\, f)(x) = 1$, so $(x, \mathrm{Re}\, f) \in \Pi(X_R)$. Given $(x, g) \in \Pi(X_R)$, there exists $f \in S(X')$ with $\mathrm{Re}\, f = g$. Since $|f(x)| \le 1$ and $(\mathrm{Re}\, f)(x) = 1$, we have $f(x) = 1$ and so $(x, f) \in \Pi(X)$.

Corollary 5. Let $T \in B(X)$, and let T_R denote T regarded as an operator on X_R. Then

$$V(T_R) = \mathrm{Re}\, V(T).$$

Proof. Immediate from Corollary 4.

The following results on upper semi-continuous set valued mappings will be useful for §20.

Definition 6. Let E, F be topological spaces, and let 2^F denote the set of all subsets of F. A mapping $\phi : E \to 2^F$ is said to be upper semi-continuous (u. s. c.) if for each $x \in E$ and each neighbourhood U of $\phi(x)$, there exists a neighbourhood V of x such that

$$y \in V \Rightarrow \phi(y) \subset U.$$

Roughly, if y is near x then all points of $\phi(y)$ are near $\phi(x)$.

For metric spaces E, F with F compact there is a convenient 'closed graph' criterion for upper semi-continuity, as follows.

Lemma 7. *Let E, F be metric spaces with F compact, let ϕ be a mapping of E into 2^F such that $\phi(x)$ is closed for each $x \in E$. Then ϕ is u. s. c. if and only if*

$$x_n \in E,\ y_n \in \phi(x_n)\ (n=1, 2, \ldots),\ x = \lim_{n\to\infty} x_n,\ y = \lim_{n\to\infty} y_n \Rightarrow y \in \phi(x).$$

Proof. Assume that ϕ is u. s. c., that $x_n \in E$, $y_n \in \phi(x_n)$, $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Let U be a closed neighbourhood of $\phi(x)$. Then there exists a neighbourhood V of x such that $\phi(z) \subset U$ $(z \in V)$. Since $x_n \in V$ for all sufficiently large n, we have $y_n \in \phi(x_n) \subset U$ for all sufficiently large n, and so $y \in U$. Thus y belongs to every closed neighbourhood of $\phi(x)$, and since $\phi(x)$ is closed this gives $y \in \phi(x)$.

Assume next that ϕ satisfies the closed graph criterion, and that U is a neighbourhood of $\phi(x)$. If there is no neighbourhood V of x satisfying $\phi(z) \subset U$ $(z \in V)$, then for every positive integer n, there exists $x_n \in E$ such that $d(x_n, x) < \frac{1}{n}$ but $\phi(x_n) \not\subset U$. Choose $y_n \in \phi(x_n)\backslash U$. Since F is compact there exists a subsequence $\{y_{n_k}\}$ with $\lim_{k\to\infty} y_{n_k} = y \in F$. Since also $\lim_{k\to\infty} x_{n_k} = x$, we conclude that $y \in \phi(x)$. Since U is a neighbourhood of $\phi(x)$ we obtain $y_{n_k} \in U$ for all sufficiently large k, a contradiction.

Lemma 8. *The mapping $x \to V(T, x)$ is an upper semi-continuous mapping of S(X) with the norm topology into the non-void compact convex subsets of $\underset{\sim}{C}$.*

Proof. The sets $V(T, x)$ are non-void compact convex subsets of a compact disc in $\mathbb{C}$, so an application of Lemma 7 will complete the proof.

Let $x_n \in S(X)$, $\lambda_n \in V(T, x_n)$, $\lim_{n\to\infty} \|x_n - x\| = 0$, $\lim_{n\to\infty} |\lambda_n - \lambda| = 0$. There exist $f_n \in D(x_n)$ with $\lambda_n = f_n(Tx_n)$. By the weak* compactness of the unit ball in X', there exists a weak* cluster point f of $\{f_n\}$ with $\|f\| \le 1$. Also

$$|1 - f(x)| \le |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)|$$
$$\le \|x_n - x\| + |(f_n - f)(x)|,$$

from which $f(x) = 1$, and so $f \in D(x)$. Finally,

$$|\lambda - f(Tx)| \le |\lambda - \lambda_n| + |f_n(Tx_n) - f_n(Tx)| + |f_n(Tx) - f(Tx)|$$
$$\le |\lambda - \lambda_n| + \|T\| \, \|x_n - x\| + |(f_n - f)(Tx)|,$$

which gives $\lambda = f(Tx) \in V(T, x)$.

We end this section with a proof of the Toeplitz-Hausdorff theorem on the convexity of the spatial numerical range of an operator on a Hilbert space. The present proof is derived from the review by Halmos [157] of a proof due to Gustafson [155]. In this review Halmos outlines a proof derived from the work of Dekker [138] on joint numerical ranges. The present proof is a modification of this using ideas that are to be found in the proof in Halmos [30], Problem 166. In what follows, H is a Hilbert space with scalar product $(\, , \,)$, and as is usual in this context the spatial numerical range of T is denoted by $W(T)$; of course we have

$$W(T) = \{(Tx, x) : x \in S(H)\}.$$

Lemma 9. *Let L be a self-adjoint element of $B(H)$, and let*

$$E = \{x \in S(H) : (Lx, x) = 0\}.$$

Then E is arcwise connected.

Proof. Note first that if $x \in E$, then $e^{i\theta}x \in E$ $(\theta \in \mathbb{R})$, and $e^{i\theta}x$

is joined to x by an arc in E. Assume then that a, b are linearly independent elements of E, choose $\theta \in \mathbb{R}$ such that $e^{i\theta}(La, b) \in i\mathbb{R}$ and take $c = e^{i\theta}a$. Then we have $(Lc, b) \in i\mathbb{R}$, and it is enough to show that we can join c to b by an arc in E. With $0 \leq \alpha \leq 1$, let $x(\alpha) = (1-\alpha)c + \alpha b$. Then

$$(Lx(\alpha), x(\alpha)) = (1-\alpha)^2(Lc, c) + (1-\alpha)\alpha\{(Lc, b)+(Lb, c)\} + \alpha^2(Lb, b),$$

and $(Lc, b) + (Lb, c) = 2\,\mathrm{Re}(Lc, b) = 0$. Thus $(Lx(\alpha), x(\alpha)) = 0$, and we obtain the required arc by taking $u(\alpha) = \|x(\alpha)\|^{-1}x(\alpha)$.

Lemma 10. Let A, B be self-adjoint elements of $B(H)$, and let

$$W = \{((Ax, x), (Bx, x)) : x \in S(H)\}.$$

Then W is a convex subset of $\mathbb{R}^2$.

Proof. It is enough to prove that $W \cap l$ is connected when l is a straight line in $\mathbb{R}^2$. Let l have equation $\alpha\xi + \beta\eta + \gamma = 0$, and let $L = \alpha A + \beta B + \gamma I$. The mapping π given by

$$\pi x = ((Ax, x), (Bx, x))$$

is continuous, and

$$E = \{x \in S(H) : (Lx, x) = 0\} = \{x \in S(H) : \pi x \in l\}.$$

Therefore $W \cap l = \pi E$ and is a connected set, by Lemma 9.

Theorem 11. (Toeplitz-Hausdorff.) Let $T \in B(H)$. Then $W(T)$ is convex.

Proof. We have $T = A + iB$ with A, B self-adjoint; and with W as in Lemma 10, $W(T) = \{\xi + i\eta : (\xi, \eta) \in W\}$.

16. THE BISHOP-PHELPS-BOLLOBÁS THEOREM

Given a non-reflexive Banach space X there is a certain lack of symmetry in the relationship between the sets $S(X)$, $S(X')$, $\Pi(X)$. For, given $x \in S(X)$ there always exists $f \in S(X')$ such that $(x, f) \in \Pi(X)$; but, given $f \in S(X')$ there need not exist any $x \in S(X)$ such that $(x, f) \in \Pi(X)$. Let us call $f \in S(X')$ a support functional if $(x, f) \in \Pi(X)$ for some x. The theorem of Bishop and Phelps [9] states that, when X is a Banach space, the set of support functionals is norm dense in $S(X')$, and we have seen already (NRI Theorem 9.4 and Theorem 10.1) that this has important implications for numerical ranges. B. Bollobás [115] has proved a stronger form of the Bishop-Phelps theorem which is even more significant for our subject, as follows.

Theorem 1. (Bishop-Phelps-Bollobás.) *Let X be a Banach space, and let* $0 < \varepsilon < 1$. *Given* $z \in X$, $h \in S(X')$ *with* $\|z\| \leq 1$ *and*

$$|1 - h(z)| < \varepsilon^2/4,$$

there exists $(y, g) \in \Pi(X)$ *such that* $\|y - z\| < \varepsilon$, $\|g - h\| < \varepsilon$.

Remarks. (1) Roughly speaking the theorem says that elements of $X \times X'$ that nearly satisfy the defining conditions for $\Pi(X)$ are close to elements of $\Pi(X)$ (in the product of the norm topologies).

(2) The Bishop-Phelps theorem is an immediate corollary. For, given $h \in S(X')$ we may choose $z \in X$ with $\|z\| \leq 1$ and $|1 - h(z)| < \varepsilon^2/4$. Then, by Theorem 1, we have a support functional $g \in X'$ with $\|g - h\| < \varepsilon$.

The proof of Theorem 1 involves only minor changes from the proof of the Bishop-Phelps theorem [9]. Note that Lemma 15.3 reduces the proof of Theorem 1 to the case when the scalar field is $\mathbb{R}$. For if X is over $\mathbb{C}$, and $|1 - h(z)| < \varepsilon^2/4$, then

$$|1 - \text{Re } h(z)| < \varepsilon^2/4.$$

Therefore if Theorem 1 has been proved for real scalars, there exist $g \in X_{\mathbb{R}}'$ with $\|g\| = 1$ and $y \in S(X)$ with $g(y) = 1$ such that $\|y - z\| < \varepsilon$ and $\|g - \text{Re } h\| < \varepsilon$. By Lemma 15.3, there exists $f \in X'$ with $g = \text{Re } f$, and we have $\|f\| = 1$, $\|f-h\| = \|g-\text{Re } h\| < \varepsilon$. Also

Re $f(y) = 1$, and $|f(y)| \leq 1$, from which $\operatorname{Im} f(y) = 0$, and $f(y) = 1$.

Notation. We suppose for the rest of this section that X is a Banach space over $\underset{\sim}{R}$ and denote by U the closed unit ball of X.

Lemma 2. Let $f, g \in S(X')$, $\varepsilon > 0$, and $T = \{x \in \frac{2}{\varepsilon}U : f(x) = 0\}$. If $|g(x)| \leq 1$ $(x \in T)$, then either $\|f-g\| \leq \varepsilon$ or $\|f+g\| \leq \varepsilon$.

Proof. Suppose that $|g(x)| \leq 1$ $(x \in T)$, and let $X_0 = \{x \in X : f(x) = 0\}$. Then $X_0 \cap U = \frac{\varepsilon}{2}T$, and so

$$\sup\{|g(x)| : x \in X_0 \cap U\} \leq \frac{\varepsilon}{2}.$$

By the Hahn-Banach theorem $g|_{X_0}$ can be extended to the whole of X without change of norm; i.e. there exists $h \in X'$ with $h|_{X_0} = g|_{X_0}$ and $\|h\| \leq \frac{\varepsilon}{2}$. Since $(g - h)(X_0) = \{0\}$, there exists $\alpha \in \underset{\sim}{R}$ with $g - h = \alpha f$, i.e.

$$g - \alpha f = h. \tag{1}$$

Suppose first that $0 \leq \alpha \leq 1$. Then

$$\begin{aligned} \|g - f\| &\leq \|g - \alpha f\| + \|(\alpha - 1)f\| \\ &= \|g - \alpha f\| + 1 - \alpha \\ &= \|g - \alpha f\| + \|g\| - \|\alpha f\| \\ &\leq 2\|g - \alpha f\| = 2\|h\| \leq \varepsilon. \end{aligned}$$

Next suppose that $\alpha > 1$. Then $0 < \frac{1}{\alpha} < 1$, and (1) takes the form

$$f - \frac{1}{\alpha}g = -\frac{1}{\alpha}h. \tag{2}$$

Comparing (1) and (2), we now have

$$\|f - g\| \leq 2\|-\frac{1}{\alpha}h\| \leq \varepsilon.$$

We have now proved that $\|g - f\| \leq \varepsilon$ if $\alpha \geq 0$. Finally if $\alpha < 0$, we consider (1) in the form

$$g - (-\alpha)(-f) = h,$$

and conclude that $\|g - (-f)\| \le \varepsilon$.

Proof of Theorem 1. We have seen already that it is enough to suppose that X is a Banach space over $\underset{\sim}{R}$.

Let $z \in U$, $h \in S(X')$, $0 < \varepsilon < 1$, $|1 - h(z)| < \varepsilon^2/4$; and take $T = \{x \in \frac{2}{\varepsilon}U : h(x) = 0\}$.

We obviously have $h(z) > 0$. Let $\kappa = \frac{1}{h(z)}(1 + \frac{2}{\varepsilon})$, and define a relation $\le$ of partial order on U by:

$$x \le y \Longleftrightarrow \|x - y\| \le \kappa h(y - x).$$

Note that $x \le y \Rightarrow h(x) \le h(y)$, since $\kappa > 0$. Let $Z = \{x \in U : z \le x\}$. We prove that Z has a maximal element y.

Given a chain (totally ordered set) $W \subset Z$, $\{h(w)\}_{w \in W}$ is an increasing net in $\underset{\sim}{R}$ bounded above by 1. Therefore $\{h(w)\}_{w \in W}$ converges. Since

$$w \le w' \Rightarrow \|w' - w\| \le \kappa(h(w') - h(w)),$$

W is a Cauchy net in the Banach space X and therefore converges to $v \in U$. By continuity of h and the norm, we have $z \le w \le v$ $(w \in W)$, and so $v \in Z$; v is an upper bound for W in Z. Therefore Z is inductively ordered, and, by Zorn's lemma, it has a maximal element y.

Since $y \in Z$, we have $z \le y$, i. e.

$$\|y - z\| \le \kappa(h(y) - h(z)).$$

Since $y \in U$, we have $h(y) \le 1$, and so

$$\begin{aligned}\|y - z\| &\le (1 + \tfrac{2}{\varepsilon})\frac{1}{h(z)}(1 - h(z)) \\ &\le (1 + \tfrac{2}{\varepsilon})(\tfrac{\varepsilon}{2})^2(1 - (\tfrac{\varepsilon}{2})^2)^{-1} \\ &= \tfrac{\varepsilon}{2}(1 - \tfrac{\varepsilon}{2})^{-1} < \varepsilon. \qquad (3)\end{aligned}$$

Let $C = co(U \cup T)$, and let p denote the Minkowski functional of C; i. e.

$$p(x) = \inf\{\alpha > 0 : \tfrac{1}{\alpha}x \in C\}.$$

Since $U \subset C$, we have

$$p(x) \leq \|x\| \qquad (x \in X).$$

We prove that $p(y) = 1$. If not, then $p(y) < 1$, since $y \in U$, and so there exists a real number α with $0 < \alpha < 1$ and $\frac{1}{\alpha} y \in C$. Therefore, U and T being convex sets, we have

$$\frac{1}{\alpha} y = \lambda u + (1 - \lambda)t$$

for some $\lambda \in [0,1]$, $u \in U$, $t \in T$. Then since $h(z) > 0$,

$$h(z) \leq h(y) = \alpha\lambda h(u) < h(u), \tag{4}$$

$$h(u - y) = (1 - \alpha\lambda)h(u) \geq (1 - \alpha\lambda)h(z). \tag{5}$$

Also $u - y = (1 - \alpha\lambda)u - \alpha(1 - \lambda)t$, and so

$$\begin{aligned} \|u - y\| &\leq (1 - \alpha\lambda) + \alpha(1 - \lambda) \|t\| \\ &\leq (1 - \alpha\lambda) + \alpha(1 - \lambda)\frac{2}{\varepsilon} \leq (1 - \alpha\lambda)(1 + \frac{2}{\varepsilon}). \end{aligned} \tag{6}$$

From (5) and (6)

$$\|u - y\| \leq \frac{1}{h(z)}(1 + \frac{2}{\varepsilon}) h(u - y) = \kappa h(u - y).$$

Therefore $y \leq u$. Since y is maximal, this gives $y = u$, which contradicts (4) and proves that $p(y) = 1$.

By the Hahn-Banach theorem, there exists a linear functional g on X with $g \leq p$ and $g(y) = p(y) = 1$. Since $y \in U$ and $p(x) \leq \|x\|$ $(x \in X)$, we have $\|y\| = 1$. Also $g \in X'$ and $\|g\| \leq 1$. Therefore

$$(y, g) \in \Pi(X).$$

By (3), $\|y - z\| < \varepsilon$.

We have $h, g \in S(X')$, and, since $T \subset C$,

$$g(x) \leq p(x) \leq 1 \qquad (x \in T).$$

Also $-T = T$, and so