MODULES OVER ENDOMORPHISM RINGS

This is an extensive synthesis of recent work in the study of endomorphism rings and their modules, bringing together direct sum decompositions of modules, the class number of an algebraic number field, point set topological spaces, and classical noncommutative localization.

The main idea behind the book is to study modules $G$ over a ring $R$ via their endomorphism ring $\text{End}_R(G)$. The author discusses a wealth of results that classify $G$ and $\text{End}_R(G)$ via numerous properties, and in particular results from point set topology are used to provide a complete characterization of the direct sum decomposition properties of $G$.

For graduate students this is a useful introduction, while the more experienced mathematician will discover that the book contains results that are not otherwise available. Each chapter contains a list of exercises and problems for future research, which provide a springboard for students entering modern professional mathematics.

THEODORE G. FATICONI is Professor in the Mathematics Department at Fordham University, New York.
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Modules over Endomorphism Rings

THEODORE G. FATICONI
Fordham University
To my wife Barbara Jean
who helped me read our book of life.
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Preface

The chapters in this book are from papers published or submitted to peer-reviewed journals. These papers were written by the author during the calendar years 2006–2008.

There is a simple example that motivates the point of view of this text. Let $k$ be a field, let $V$ be an $n$-dimensional $k$-vector space for some integer $n > 0$, and let $E = \text{Mat}_n(k)$ denote the ring of $n \times n$-matrices over $k$. Fix an ordered basis $\beta$ for $V$ and let $[v]_\beta$ denote the vector representation for $v$ relative to $\beta$. Given $r \in E$ and $v \in V$ then we define

$$rv = r \cdot [v]_\beta,$$

where $\cdot$ is the usual multiplication between the $n \times n$ matrix $r$ and the column vector $[v]_\beta$. This multiplication makes $V$ a left $E$-module. Given a right ideal $I \subset E$ we define

$$IV = \left\{ \sum_i r_i v_i \mid \text{finitely many elements } r_i \in I \text{ and } v_i \in V \right\}.$$

Then the assignment

$$I \mapsto IV$$

defines a bijection between the set of right ideals of $E$ and the set of $k$-subspaces of $V$. Thus we can study some properties of $V$ by studying the right ideals in the ring $E$. Notice that we have passed from a strictly additive setting into a setting that is additive and multiplicative. This gain in structure improves our chances of solving certain problems concerning $V$.

There is little hope of generalizing the bijection $I \mapsto IV$ to more general modules over associative rings without sacrificing something, so we hope for the best possible generalization. To find this generalization we will use elements from ring theory, module theory, and some elementary homology and homotopy theory of complexes over associative rings, and in at least a couple of instances we use some point set topology. More details follow.
Most modern research into direct sum decompositions of reduced torsion-free finite rank abelian groups (now called rtffr groups) begins with a study of projective modules over End(G). This stems from the Arnold–Lady theorem, which shows that direct summands of $G^n$ correspond to projective direct summands of End(G).

This method begins a study of modules G over a ring R and the endomorphism ring End$_R$(G). We begin by constructing a category G-plex of what are called G-plexes. This category is category equivalent to Mod-End$_R$(G), which makes G-plex the category to be studied if we wish to characterize G or End$_R$(G). A duality is used to characterize the rings End$_R$(G) whose properties are on the following list of properties of rings:

1. right or left hereditary
2. right or left Noetherian
3. right or left coherent
4. right or left FP-injective
5. right or left self-injective
6. right or left cogenerator
7. right PF rings
8. QF rings

We consider End$_R$(G) and the left End$_R$(G)-module G, and we characterize several integers associated with rings and modules. Specifically we characterize the integers on the following list:

1. projective dimension of G
2. injective dimension of G
3. flat dimension of G
4. right or left global dimension of End$_R$(G)

Several properties are left as exercises.

One of the purposes of this book is to show that we can study groups locally isomorphic to G by studying invertible fractional right ideals of $E(G)$ where

$$E(G) = \text{End}(G)/N(\text{End}(G)).$$

For example, let $n > 0$ be an integer, and given a commutative prime ring R, let Pic(R) denote the abelian group of isomorphism classes of invertible fractional right ideals of R. If G is a strongly indecomposable rtffr group and if $E(G)$ is commutative then the set of isomorphism classes (H) of groups H that are locally isomorphic to $G^n$ is bijective with the finite abelian group Pic($E(G)$). In this setting we show that if $H$, $K$, and $L$ are direct summands of $G^n$, then cancellation in the isomorphism $H \oplus K \cong H \oplus L$ can be viewed as cancellation of elements in the abelian group Pic($E(G)$).

This point of view gives a new insight into the problem of finding the class number $h(k)$ of the algebraic number field k. For example, those k with $h(k) = 1$ are classified
in the following result. Let $\mathbb{E}$ denote the algebraic integers in $k$. Let $\Omega(\mathbb{E}) = \{\text{rtffr groups } G \mid \text{End}(G) \cong \mathbb{E}\}$. 

**Theorem.** The following are equivalent for the algebraic number field $k$.

1. $h(k) = 1$.
2. Each $G \in \Omega(k)$ has the power cancellation property. (For each integer $m > 0$ and group $H$, $G^m \cong H^m$ implies that $G \cong H$.)
3. Each group $H$ that is locally isomorphic to $G$ is isomorphic to $G$.

Some research over the thirty-year period from 1970 to 2000 dealt with the rtffr groups $G$ that were finitely generated left $\text{End}(G)$-modules, or projective left $\text{End}(G)$-modules, or that had right hereditary endomorphism ring. Our thirty-odd pages on this type of result give us a unified approach to these problems and extends existing results. Subsequently, we use the machinery developed in Chapter 9 to characterize the left $\text{End}(G)$-module $G$ that possesses some properties from the following list:

1. finitely generated
2. finitely presented
3. coherent
4. projective
5. quasi-projective
6. possesses a projective cover
7. cogenerator
8. generator
9. progenerator
10. quasi-progenerator
11. Noetherian

Let $R$ be an associative ring with identity, let $G$ be a right $R$-module, and let $\text{End}_R(G)$ denote the ring of $R$-endomorphisms of $G$. The module $G$ is *self-small* if for each index set $\mathcal{I}$ and each $R$-module map $\phi : G \longrightarrow G^{(\mathcal{I})}$ there is a finite set $\mathcal{J} \subset \mathcal{I}$ such that $\phi(G) \subset G^{(\mathcal{J})}$. In other words there is a natural isomorphism

$$\text{Hom}_R(G, G^{(\mathcal{I})}) \longrightarrow \text{Hom}_R(G, G^{(\mathcal{J})}).$$

Let $P(G) = \{\text{right } R\text{-modules } Q \mid Q \oplus Q' \cong G^{(\mathcal{I})} \text{ for some index set } \mathcal{I} \text{ and some right } R\text{-module } Q'\}$. A $G$-plex is a complex

$$Q = \cdots \longrightarrow Q_3 \xrightarrow{\delta_2} Q_2 \xrightarrow{\delta_1} Q_1 \xrightarrow{\delta_1} Q_0$$
with the properties that

1. $Q_k \in \mathbf{P}(G)$ for each $k \geq 0$ and
2. $G$ has the following lifting property for each $k \geq 1$. Given a map $\phi : G \to Q_k$ such that $\delta_k \phi = 0$ there is a map $\psi : G \to Q_{k+1}$ such that $\phi = \delta_{k+1} \psi$ as in the commutative triangle

$$
\begin{array}{ccc}
Q_{k+1} & \xrightarrow{\delta_{k+1}} & Q_k \\
\downarrow & & \downarrow \psi \\
Q_k & \xrightarrow{\delta_k} & Q_{k-1}
\end{array}
$$

of right $R$-modules.

The category of $G$-plexes $G$-Plex is the additive category whose objects are the $G$-plexes $Q$ and whose morphisms are homotopy equivalence classes $[f]$ of chain maps

$$f : Q \to Q'$$

between $G$-plexes $Q$ and $Q'$.

If we let $\text{Mod-End}_R(G)$ denote the category of right $\text{End}_R(G)$-modules then the functor

$$h_G(\cdot) : G$\text{-Plex} \to \text{Mod-End}_R(G)$$

sends $Q \in G$-Plex to the zeroth homology group of the complex

$$\text{Hom}(G, Q) = \cdots \xrightarrow{\delta^+_2} \text{Hom}_R(G, Q_1) \xrightarrow{\delta^+_1} \text{Hom}_R(G, Q_0),$$

or in other words

$$h_G(Q) = \text{coker} \, \delta^+_1.$$

**Theorem.** Let $G$ be a self-small right $R$-module. Then the additive functor

$$h_G(\cdot) : G$\text{-Plex} \to \text{Mod-End}_R(G)$$

is a category equivalence.

Thus the category of right $\text{End}_R(G)$-modules, $\text{Mod-End}_R(G)$, is characterized in terms of a category $G$-Plex in which $G$ is a small projective generator.
One of the more attractive elements of this point of view is that it dualizes without too much effort. We will assume the set theoretic condition
\[(\mu)\text{ measurable cardinals do not exist.}\]
The assumption \((\mu)\) is true under Gödel’s constructibility hypothesis. Under \((\mu)\) we can make a complete dualization of the above theorem. The right \(R\)-module \(G\) is self-slender if for each index set \(I\) and \(R\)-module map \(\phi : G^I \rightarrow G\) there is a finite set \(J \subset I\) such that
\[G^{I \setminus J} \subset \ker \phi.\]
Equivalently \(G\) is self-slender if for each index set \(I\) the canonical map
\[
\text{Hom}_{R}(G, G)^{(I)} \rightarrow \text{Hom}_{R}(G^I, G)
\]
is an isomorphism. Let
\[
W = W_0 \xrightarrow{\sigma_1} W_1 \xrightarrow{\sigma_2} W_2 \xrightarrow{\sigma_3} \cdots
\]
be a complex of right \(R\)-modules. Then \(W\) is a \(G\)-coplex if \(W_k\) is a direct summand of a direct product of copies of \(G\) for each integer \(k \geq 0\), and if it satisfies the lifting property that is dual to the lifting property satisfied by a \(G\)-plex. Define the category of \(G\)-coplexes, \(G\text{-Coplx}\), to be that category whose objects are \(G\)-coplexes and whose maps are homotopy equivalence classes \([f]\) of chain maps \(f\) between \(G\)-coplexes. The functor
\[
h^G(\cdot) : G\text{-Coplx} \rightarrow \text{End}_{R}(G)\text{-Mod}
\]
is defined by
\[
h^G(W) = \text{coker Hom}_{R}(\partial_1, G)
\]
which is just the zeroth homology group of the complex of left \(\text{End}_{R}(G)\)-modules \(\text{Hom}_{R}(W, G)\).

**Theorem.** Assume \((\mu)\) and let \(G\) be a self-slender right \(R\)-module. Then the additive functor
\[
h^G(\cdot) : G\text{-Coplx} \rightarrow \text{End}_{R}(G)\text{-Mod}
\]
is a category equivalence.

Consequently, we have characterized the category \(\text{End}_{R}(G)\text{-Mod}\) of \(\text{left \ End}_{R}(G)\)-modules in terms of the category \(G\text{-Coplx}\) in which \(G\) is a slender injective cogenerator. It is worth noting that if \(G\) is a reduced torsion-free finite rank
abelian group then $G$ is both self-small and self-slender. Thus for these groups we have a complete characterization of the right $\text{End}_R(G)$-modules and the left $\text{End}_R(G)$-modules by categories completely determined by $G$.

From these theorems we sample the existing module theoretic properties for $G$ that can be characterized in terms of $\text{End}_R(G)$, and we look at those properties of $\text{End}_R(G)$ that can be characterized in terms of $G$. Specifically we characterize the homological dimensions of $G$ as a left $\text{End}_R(G)$-module, we characterize the global dimensions of $\text{End}_R(G)$ in terms of $G$, and we consider ring theoretic properties for $\text{End}_R(G)$. E.g. we determine when $\text{End}_R(G)$ is left or right Noetherian, left or right coherent, right or left self-injective, a left or right cogenerator ring, a left or right PF ring, a $QF$ ring, or a left or right $FP$-injective ring. We also characterize those $C$ such that $\text{Hom}_R(G, C)$ is a projective or an injective right $\text{End}_R(G)$-module.

There are a few diagrams that illustrate a connection between $G$, $\text{End}_R(G)$, homology, and point set topological spaces. This equivalence of ideas from different areas of mathematics is rare. Given a (not necessarily self-small) right $R$-module $G$ there is a commutative diagram of categories and functors in which $\text{M-spaces}$ denotes a category whose objects are point set topological spaces with specified homology groups. It is usual to call a space concentrated at some integer $k \geq 0$ a Moore $k$-space. Notice that the diagram 16.1 contains the left modules and the right modules over $\text{End}_R(G)$, as well as the functors $\text{Tor}^*$ and $\text{Ext}^*$.

Fix $G$. By letting $X$ denote the topological space whose fundamental group is $G$, called an Eilenberg–MacLane space, we develop a commutative triangle (see Diagram 17.1) of categories and functors. When $G$ is self-small this triangle consists of category equivalences between homology theory, modules over a ring, and a category of point set topology.
The text ends with a chapter that combines noncommutative localization of rings with an additive functor $QH_G(\cdot)$, and a new measurement of modules called margimorphism to give several right $R$-modules $G$ possessing unique decompositions. For example, let $Q_G$ denote the semi-primary classical right ring of quotients of the ring $\text{End}_R(G)$. Then $G$ is margimorphic to $G'$ iff $QH_G(G) \cong QH_G(G')$ as right $Q_G$-modules. Furthermore, we prove that $Q_G$ can be used to find a unique direct sum decomposition for $G$.

**Theorem.** Suppose that $\text{End}_R(G)$ possesses a semi-primary classical right ring of quotients $Q_G$ such that $Q_G/I(Q_G)$ is a product of division rings. Then $G$ has a unique direct sum decomposition in the sense of the Azumaya–Krull–Schmidt theorem.

**Organization:** Aside from a preliminary chapter, the book is in three parts:

1. A portion of the book is devoted to the study of a number-theoretic connection between $G$ and $E(G)$. This includes an investigation into the algebraic number theory of algebraic number fields.
2. A portion of the book is devoted to the study of the module and ideal theoretic connections between $G$ and $\text{End}(G)$.
3. A portion of the book shows a categorical connection between $G$, $\text{End}(G)$, and certain point set topological spaces.

Chapters 2–7 develop a method for utilizing the commutative property in $E(G)$ in discussing unique direct sum decompositions of $G$. These techniques characterize the class number of an algebraic number field. Chapter 8 uses analytic number theory to study groups $G$ such that each group locally isomorphic to $G$ is isomorphic to $G$. Chapter 9 gives the homological framework needed to study $\text{End}(G)$ systematically, including the theorem giving the category equivalence $h_G : G\text{-Plex} \to \text{Mod-End}_R(G)$. Chapter 10 gives several hypotheses under which the tensor functor $T_G$ with $G$ is a category equivalence. In Chapter 11 we describe the ring $\text{End}_R(G)$ with small right or left global dimension. Chapters 12–15 give characterizations of modules of the form $\text{Hom}_R(G, C)$. Chapters 16 and 17 give the diagrams relating abelian groups, modules, $\text{End}_R(G)$, and point set topology. These techniques characterize the class number of an algebraic number field. Chapter 18 is devoted to margimorphisms.
Each chapter ends with a number of exercises and the chapters themselves contain many statements of the type *the reader will prove that* . . . These are details or generalizations that I felt detracted from the discussion. The young ring or module theorist should attempt these exercises. Since examples guide our intuition and guide us to theorems, the reader should not be surprised at the number of examples used to motivate our discussions.

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Since my research style produces many \TeX files and almost no paper files, I am both author and technical typist on this project. Any errors within are my responsibility.