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978-0-521-19855-4 — The Geometrical Language of Continuum Mechanics
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# 1 The Case for Differential Geometry

If Mathematics is the language of Physics, then the case for the use of Differential Geometry in Mechanics needs hardly any advocacy. The very arena of mechanical phenomena is the *space-time continuum*, and a continuum is another word for a *differentiable manifold*. Roughly speaking, this foundational notion of Differential Geometry entails an entity that can support *smooth fields*, the physical nature of which is a matter of context. In Continuum Mechanics, as opposed to Classical Particle Mechanics, there is another continuum at play, namely, the *material body*. This continuous collection of particles, known also as the *body manifold*, supports fields such as temperature, velocity and stress, which interact with each other according to the physical laws governing the various phenomena of interest. Thus, we can appreciate how Differential Geometry provides us with the proper mathematical framework to describe the two fundamental entities of our discourse: the space-time manifold and the body manifold. But there is much more.

When Lagrange published his treatise on analytical mechanics, he was in fact creating, or at least laying the foundations of, a Geometrical Mechanics. A classical mechanical system, such as the plane double pendulum shown in Figure 1.4, has a finite number of degrees of freedom. In this example, because of the constraints imposed by the constancy of the lengths of the links, this number is 2. The first mass may at most describe a circumference with a fixed centre, while the second mass may at most describe a circumference whose centre is at the instantaneous position of the first mass. As a result, the surface of a torus emerges in this example as the descriptor of the *configuration space* of the system. Not surprisingly, this configuration space is, again, a differentiable manifold. This notion escaped Lagrange, who regarded the degrees of freedom as coordinates, without asking the obvious question: coordinates of what? It was only later, starting with the work of Riemann, that the answer to this question was clearly established. The torus, for example (or the surface of a sphere, for that matter), cannot be covered with a single coordinate system. Moreover, as Lagrange himself knew, the choice of coordinate systems is quite arbitrary. The underlying geometrical object, however, is always one and the same. This distinction between the underlying geometrical (and physical) entity, on the one hand, and the coordinates used to represent it, on the other hand, is one of the essential features of modern Differential Geometry. It is also an essential feature of modern Physics. The formulation of physical principles, such as the principle of virtual power, may attain a high degree of simplicity when expressed in geometrical terms. When moving into the realm of Continuum Mechanics, the situation gets complicated by the fact that the number of degrees of freedom of a continuous system is infinite. Nevertheless, at least in principle, the geometric picture is similar.

There is yet another aspect, this time without a Particle Mechanics counterpart, where Differential Geometry makes a natural appearance in Continuum Mechanics. This is the realm of *constitutive equations*. Whether because of having smeared the molecular interactions, or because of the need to agree with experimental results at a macroscopic level in a wide variety of materials, or for other epistemological reasons, the physical laws of Continuum Mechanics do not form a complete system. They need to be supplemented with descriptors of material response known as *constitutive laws* expressed in terms of constitutive equations. When seen in the context of infinite dimensional configuration spaces, as suggested above, the constitutive equations themselves can be regarded as geometric objects. Even without venturing into the infinite-dimensional domain, it is a remarkable fact that the specification of the constitutive equations of a material body results in a well-defined differential geometric structure, a sort of *material geometry*, whose study can reveal the presence of continuous distributions of material defects or other kinds of *material inhomogeneity*.

In the remainder of this motivational chapter, we will present in a very informal way some basic geometric differential concepts as they emerge in appropriate physical contexts. The concept of differentiable manifold (or just *manifold*, for short) will be assumed to be available, but we will content ourselves with the mental picture of a continuum with a definite dimension.[1] Not all the motivational lines suggested in this chapter will be pursued later in the book. It is also worth pointing out that, to this day, the program of a fully fledged geometrization of Continuum Mechanics cannot be said to have been entirely accomplished.

## 1.1. Classical Space-Time and Fibre Bundles

### 1.1.1. Aristotelian Space-Time

We may think separately of time as a 1-dimensional manifold $\mathcal{Z}$ (the time line) and of space as a 3-dimensional manifold $\mathcal{P}$. Nevertheless, as soon as we try to integrate these two entities into a single space-time manifold $\mathcal{S}$, whose points represent *events*, we realize that there are several possibilities. The first possibility that comes to mind is what we may call *Aristotelian space-time*, whereby time and space have independent and absolute meanings. Mathematically, this idea corresponds to the product:

$$\mathcal{S}_A = \mathcal{Z} \times \mathcal{P}, \tag{1.1}$$

[1] The rigorous definition of a manifold will be provided later in Chapter 4.

where $\times$ denotes the Cartesian product. Recall that the *Cartesian product* of two sets is the set formed by all ordered pairs such that the first element of the pair belongs to the first set and the second element belongs to the second set. Thus, the elements $s$ of $\mathcal{S}_A$, namely, the events, are ordered pairs of the form $(t,p)$, where $t \in \mathcal{Z}$ and $p \in \mathcal{P}$. In other words, for any given $s \in \mathcal{S}_A$, we can determine independently its corresponding temporal and spatial components. In mathematical terms, we say that the 4-dimensional (product) manifold $\mathcal{S}_A$ is endowed with two *projection maps*:

$$\pi_1 : \mathcal{S}_A \longrightarrow \mathcal{Z}, \tag{1.2}$$

and

$$\pi_2 : \mathcal{S}_A \longrightarrow \mathcal{P}, \tag{1.3}$$

defined, respectively, by:

$$\pi_1(s) = \pi_1(t,p) := t, \tag{1.4}$$

and

$$\pi_2(s) = \pi_2(t,p) := p. \tag{1.5}$$

### 1.1.2. Galilean Space-Time

The physical meaning of the existence of these two natural projections is that any observer can tell independently whether two events are simultaneous and whether or not (regardless of simultaneity) they have taken place at the same location in space. According to the principle of *Galilean relativity*, however, this is not the case. Two different observers agree, indeed, on the issue of simultaneity. They can tell unequivocally, for instance, whether or not two light flashes occurred at the same time and, if not, which preceded which and by how much. Nevertheless, in the case of two nonsimultaneous events, they will in general disagree on the issue of position. For example, an observer carrying a pulsating flashlight will interpret successive flashes as happening always "here," while an observer receding uniformly from the first will reckon the successive flashes as happening farther and farther away as time goes on. Mathematically, this means that we would like to get rid of the nonphysical second projection (the spatial one) while preserving the first projection.

We would like, accordingly, to construct an entity that looks like $\mathcal{S}_A$ for each observer, but which is a different version of $\mathcal{S}_A$, so to speak, for different observers. This delicate issue can be handled as follows. We define space-time as a 4-dimensional manifold $\mathcal{S}$ endowed with a projection map:

$$\pi : \mathcal{S} \longrightarrow \mathcal{Z}, \tag{1.6}$$

together with a collection of smooth and (smoothly) invertible maps:

$$\phi : \mathcal{S} \longrightarrow \mathcal{S}_A, \tag{1.7}$$

onto the naive Aristotelian space-time $\mathcal{S}_A$. Each of these maps, called a *trivialization* and potentially representing an observer, cannot be completely arbitrary, in a sense that we will now explain.

Fix a particular point of time $t \in \mathcal{Z}$ and consider the inverse image $\mathcal{S}_t = \pi^{-1}(\{t\})$. We call $\mathcal{S}_t$ the *fibre of $\mathcal{S}$ at $t$*. Recall that the *inverse image* of a subset of the range of a function is the collection of all the points in its domain that are mapped to points in that subset. With this definition in mind, the meaning of $\mathcal{S}_t$ is the collection of all events that may happen at time $t$. We clearly want this *collection* to be the same for all observers, a fact guaranteed by the existence of the projection map $\pi$. Different observers will differ only in that they will attribute possibly different locations to events in this fibre. Therefore, we want the maps $\phi$ to be *fibre preserving* in the sense that each fibre of $\mathcal{S}$ is mapped to one and the same fibre in $\mathcal{S}_A$. In other words, we don't want to mix in any way whatsoever the concepts of space and time. We require, therefore, that the image of each fibre in $\mathcal{S}$ by each possible $\phi$ be exactly equal to a fibre of $\mathcal{S}_A$. More precisely, for each $t \in \mathcal{Z}$ we insist that:

$$\phi(\mathcal{S}_t) = \pi_1^{-1}(\{t\}). \tag{1.8}$$

A manifold $\mathcal{S}$ endowed with a projection $\pi$ onto another manifold $\mathcal{Z}$ (called the *base manifold*) and with a collection of smooth invertible fibre-preserving maps onto a product manifold $\mathcal{S}_A$ (of the base times another manifold $\mathcal{P}$) is known as a *fibre bundle*. Note that the fibres of $\mathcal{S}_A$ by $\pi_1$ are all exact copies of $\mathcal{P}$. We say that $\mathcal{P}$ is the *typical fibre* of $\mathcal{S}$. A suggestive pictorial representation of these concepts is given in Figure 1.1. A more comprehensive treatment of fibre bundles will be presented in Chapter 7.

EXAMPLE 1.1. **Microstructure:** A completely different application of the notion of fibre bundle is the description of bodies with internal structure, whereby the usual kinematic degrees of freedom are supplemented with extra degrees of freedom intended to describe a physically meaningful counterpart. This idea, going at least as far back as the work of the Cosserat brothers,[2] applies to diverse materials, such as liquid crystals and granular media. The base manifold represents the matrix, or *macromedium*, while the fibres represent the *micromedium* (the elongated molecules or the grains, as the case may be).

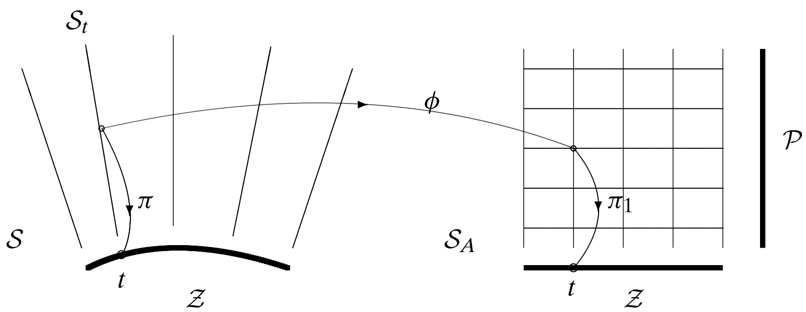


Figure 1.1. A fibre bundle

[2] Cosserat E, Cosserat F (1909), *Théorie des corps déformables*, Hermann et Fils, Paris.

Notice in Figure 1.1 how the fibres are shown hovering above (rather than touching) the base manifold. This device is used to suggest that, although each fibre is assigned to a specific point of the base manifold, the fibre and the base do not have any points in common, nor is there any preferential point in the fibre (such as a zero). Quite apart from the ability of Differential Geometry to elicit simple mental pictures to describe very complex objects, such as a fibre bundle, another important feature is that it uses the minimal amount of structure necessary. In the case of the space-time bundle, for instance, notice that we have not made any mention of the fact that there is a way to measure distances in space and a way to measure time intervals. In other words, what we have presented is what might be called a *proto-Galilean space-time*, where the notion of simultaneity has a physical (and geometrical) meaning. Beyond that, we are now in a position to impose further structure either in the base manifold, or in the typical fibre, or in both. Similarly, restrictions can be placed on the maps $\phi$ (governing the change of observers). In classical *Galilean space-time*, the fibre $\mathcal{P}$ has the structure of an *affine space* (roughly a vector space without an origin). Moreover, this vector space has a distinguished *dot product*, allowing to measure lengths and angles. Such an affine space is called a *Euclidean space*. The time manifold $\mathcal{Z}$ is assumed to have a Euclidean structure as well, albeit 1-dimensional. Physically, these structures mean that there is an observer-invariant way to measure distances and angles in space (at a given time) and that there is also an observer-invariant way to measure intervals of time. We say, accordingly, that Galilean space-time is an *affine bundle*. In such a fibre bundle, not only the base manifold and the typical fibre are affine spaces, but also the functions $\phi$ are limited to affine maps. These are maps that preserve the affine properties (for example, parallelism between two lines). In the case of Euclidean spaces, the maps may be assumed to preserve the metric structure as well.

### 1.1.3. Observer Transformations

Having identified an observer with a trivialization $\phi$, we can consider the notion of *observer transformation*. Let $\phi_1 : \mathcal{S} \to \mathcal{S}_A$ and $\phi_2 : \mathcal{S} \to \mathcal{S}_A$ be two trivializations. Since each of these maps is, by definition, invertible and fibre preserving, the composition:

$$\phi_{1,2} = \phi_2 \circ \phi_1^{-1} : \mathcal{S}_A \to \mathcal{S}_A, \tag{1.9}$$

is a well-defined fibre-preserving map from $\mathcal{S}_A$ onto itself. It represents the transformation from observer number 1 to observer number 2. Because of fibre preservation, the map $\phi_{1,2}$ can be seen as a smooth collection of time-dependent maps $\tilde{\phi}^t_{1,2}$ of the typical fibre $\mathcal{P}$ onto itself, as shown schematically in Figure 1.2. In Galilean space-time proper, we limit these maps to affine maps that preserve the orientation and the metric (Euclidean) structure of the typical fibre $\mathcal{P}$ (which can be seen as the usual 3-dimensional Euclidean space).

Among all such maps $\tilde{\phi}^t_{1,2} : \mathcal{P} \to \mathcal{P}$, it is possible to distinguish some that not only preserve the Euclidean structure but also represent changes of observers that travel with respect to each other at a fixed inclination (i.e., without angular velocity) and at

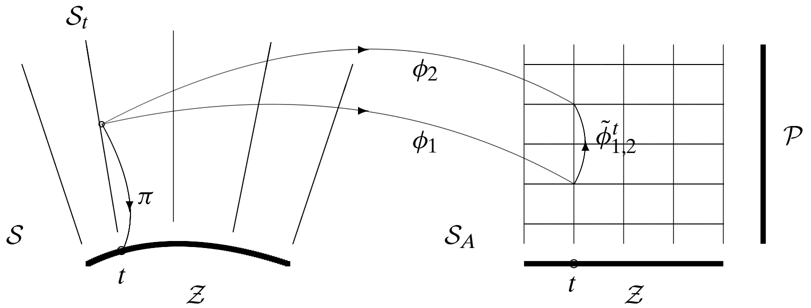


Figure 1.2. Observer transformation

a constant velocity of relative translation. Observers related in this way are said to be *inertially related*. It is possible, accordingly, to divide the collection of all observers into equivalence classes of inertially related observers. Of all these inertial classes, Isaac Newton declared one to be privileged above all others.[3] This is the class of *inertial observers*, for which the laws of Physics acquire a particularly simple form.

### 1.1.4. Cross Sections

A *cross section* (or simply a *section*) of a fibre bundle $\mathcal{S}$ is a map $\Gamma$ of the base manifold $\mathcal{Z}$ to the fibre bundle itself:

$$\Gamma : \mathcal{Z} \to \mathcal{S}, \tag{1.10}$$

with the property:

$$\pi \circ \Gamma = id_{\mathcal{Z}}, \tag{1.11}$$

where $id_{\mathcal{Z}}$ is the identity map of the base manifold and where "$\circ$" denotes the composition of maps. This property expresses the fact that the image of each element of the base manifold is actually in the fibre attached to that element. A convenient way to express this fact is by means of the following commutative diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\Gamma"] \arrow[dr, "id_{\mathcal{Z}}"'] & \mathcal{S} \arrow[d, "\pi"] \\ & \mathcal{Z} \end{tikzcd} \tag{1.12}$$

Pictorially, as shown in Figure 1.3, the image of a section looks like a line cutting through the fibres, hence its name. For general fibre bundles, there is no guarantee that a smooth (or even continuous) cross section exists.

In the case of Galilean space-time, a section represents a *world line* or, more classically, a *trajectory* of a particle.

[3] This appears to be the meaning of Newton's first law of motion.

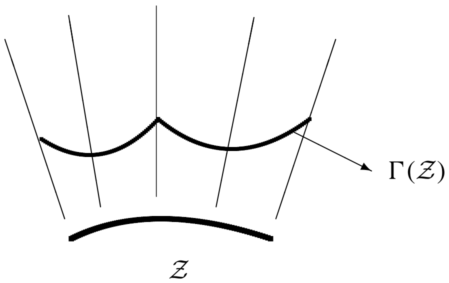


Figure 1.3. A (not necessarily smooth) section

EXERCISE 1.1. Explain on physical grounds why a world-line must be a cross section, that is, it cannot be anywhere tangent to a (space) fibre of space-time.

EXERCISE 1.2. How does the world-line of a particle at rest look in Aristotelian space-time? Let $\phi : \mathcal{S} \to \mathcal{S}_A$ be a trivialization of Galilean space-time, and let $\sigma : \mathcal{Z} \to \mathcal{S}_A$ be a constant section of Aristotelian space-time. Write an expression for the world-line of a particle at rest with respect to the observer defined by $\phi$. Do constant sections exist in an arbitrary fibre bundle?

EXERCISE 1.3. Draw schematically a world diagram for a collision of two billiard balls. Comment on smoothness.

EXERCISE 1.4. How does the motion of a cloud of particles look in space-time? Justify the term "world-tube" for the trajectory of a continuous medium of finite extent.

### 1.1.5. Relativistic Space-time

The revolution brought about by the theory of Relativity (both in its special and general varieties) can be said to have destroyed the bundle structure altogether. In doing so, it in fact simplified the geometry of space-time, which becomes just a 4-dimensional manifold $\mathcal{S}_R$. On the other hand, instead of having two separate metric structures, one for space and one for time, Relativity assumes the existence of a space-time metric structure that involves both types of variables into a single construct. This type of metric structure is what Riemann had already considered in his pioneering work on the subject, except that Relativity (so as to be consistent with the Lorentz transformations) required a metric structure that could lead both to positive and to negative squared distances between events, according to whether or not they are reachable by a ray of light. In other words, the metric structure of Relativity is not positive definite. By removing the bundle structure of space-time, Relativity was able to formulate a geometrically simpler picture of space-time, although the notion of simplicity is in the eyes of the beholder.

## 1.2. Configuration Manifolds and Their Tangent and Cotangent Spaces

### 1.2.1. The Configuration Space

We have already introduced the notion of configuration space of a classical mechanical system and illustrated it by the example of a plane double pendulum. Figure 1.4 is a graphical representation of what we had in mind.

The coordinates $\theta_1$ and $\theta_2$, an example of so-called *generalized coordinates*, cannot be used globally (i.e., over the entire torus) since two configurations that differ by $2\pi$ in either coordinate must be declared identical. Moreover, other coordinate choices are also possible (for example, the horizontal deviations from the vertical line at the point of suspension). Each point on the surface of the torus corresponds to one configuration, and vice versa. The metric properties of the torus are not important. What matters is its *topology* (the fact that it looks like a doughnut, with a hole in the middle). The torus is, in this case, the *configuration space* (or *configuration manifold*) $\mathcal{Q}$ of the dynamical system at hand.

EXERCISE 1.5. Describe the configuration space of each of the following systems:

(1) A free particle in space.
(2) A rigid bar in the plane.
(3) An inextensible pendulum in the plane whose point of suspension can move along a rail.
(4) A pendulum in space attached to a fixed support by means of an inextensible string that can wrinkle.

### 1.2.2. Virtual Displacements and Tangent Vectors

We are now interested in looking at the concept of *virtual displacement*. Given a configuration $q \in \mathcal{Q}$, we consider a small perturbation to arrive at another, neighbouring, configuration, always moving over the surface of the torus (since the system cannot

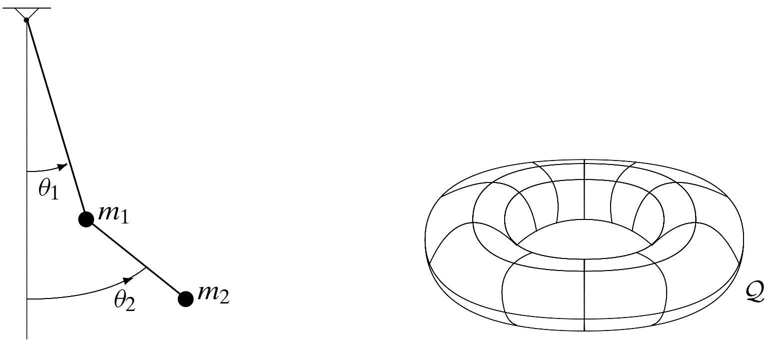


Figure 1.4. The plane double pendulum and its configuration manifold

escape the trap of its own configuration space). Intuitively, what we have is a small piece of a curve in $\mathcal{Q}$, which we can approximate by a *tangent vector*.

To make this notion more precise, imagine that we have an initially unstretched thin elastic ruler on which equally spaced markers have been drawn, including a zero mark. If we now stretch or contract this ruler, bend it and then apply it to the surface of the torus at some point $q$, in such a way that the zero mark falls on $q$, we obtain an intuitive representation of a *parametrized curve* $\gamma$ on the configuration manifold. Let us now repeat this procedure ad infinitum with all possible amounts of bending and stretching, always applying the deformed ruler with its zero mark at the same point $q$. Among all the possible curves obtained in this way, there will be a subcollection that shares the same tangent and the same stretch with $\gamma$. We call this whole collection (technically known as an *equivalence class* of parametrized curves) a *tangent vector to the configuration manifold* at $q$. Notice that, although when we draw this tangent vector $\mathbf{v}$ in the conventional way as an arrow, it seems to contradict the fact that we are supposed to stay on the surface, the definition as an equivalence class of curves (or, less precisely, a small piece of a curve) removes this apparent contradiction. Any of the curves in the equivalence (e.g., the curve $\gamma$ of departure) can be used as the *representative* of the vector. The vector can also be regarded as a *derivation* with respect to the curve parameter (the equally spaced markers).

The collection of all tangent vectors at a point $q \in \mathcal{Q}$ is called the *tangent space of* $\mathcal{Q}$ *at* $q$ and is denoted by $T_q\mathcal{Q}$. In the case of the torus, the interpretation of $T_q\mathcal{Q}$ is the tangent plane to the torus at $q$, as shown in Figure 1.5. The tangent space at a point $q$ of the configuration space is the carrier of all the possible virtual displacements away from the configuration represented by $q$. A physically appealing way to look at virtual displacements is as *virtual velocities* multiplied by a small time increment.

### 1.2.3. The Tangent Bundle

We now venture into a further level of abstraction. Assume that we attach to each point its tangent space (just like one would attach a small paper sticker at each point of a globe). We obtain a collection denoted by $T\mathcal{Q}$ and called the *tangent bundle* of $\mathcal{Q}$. An element of this contraption consists of a point plus a tangent vector attached to it

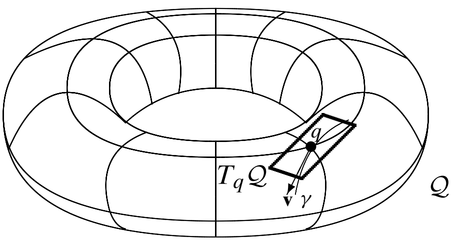


Figure 1.5. A tangent space

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(i.e., a configuration and a virtual displacement away from it). It requires, therefore, four parameters: the two coordinates of the point and the two components of the vector with respect to some basis of the tangent space. Since everything is smooth, it is not difficult to imagine that we are now in the presence of a 4-dimensional manifold $T\mathcal{Q}$. Moreover, given a tangent vector (a sticker, say), we can unambiguously say to which point it is attached. In other words, we a have a natural *projection map* $\tau$:

$$\tau : T\mathcal{Q} \to \mathcal{Q}, \tag{1.13}$$

Thus, the tangent bundle $T\mathcal{Q}$ has the natural structure of a fibre bundle. Its dimension is always equal to twice the dimension of its base manifold $\mathcal{Q}$. Its typical fibre is an $n$-dimensional vector space. Since all finite dimensional vector spaces of the same dimension are equivalent to each other, we may say that the typical fibre of the tangent bundle of an $n$-dimensional manifold is $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$.

A section of the tangent bundle is a map $\Gamma : \mathcal{Q} \to T\mathcal{Q}$ satisfying condition (1.11), namely:

$$\tau \circ \Gamma = id_{\mathcal{Q}}. \tag{1.14}$$

The physical meaning of a section of the tangent bundle is a *vector field*, since at each point $q$ of $\mathcal{Q}$ the map $\Gamma$ appoints a tangent vector $\Gamma(q)$.

### 1.2.4. The Cotangent Bundle

We move one step further into the physics and the geometry of the situation. Consider the notion of a linear operator on virtual displacements. This entity would be a black box that produces real numbers whenever it receives as an input a virtual displacement, just as a slot machine produces a pack of gum whenever you insert a coin. The fact that we are looking at a *linear* operator means that multiplying the input by a number results in the output being multiplied by the same number. Moreover, an input equal to the sum of two other inputs elicits an output exactly equal to the sum of the outputs elicited by the inputs acting separately. If the inputs are virtual displacements, the physical entity which behaves just like that is nothing but a *force*! And the corresponding output of our linear machine is nothing but the *virtual work* of the force over the given virtual displacement. Thus, forces are not vectors that cohabit with the virtual displacements upon which they act. They are vectors, indeed, in the sense that they also can be added and multiplied by numbers. They, however, live in their own vector space of linear operators and manifest themselves by the result of their (linear) action on the virtual displacement vectors. The collection of linear operators over the vectors of a vector space $V$ is known as the *dual space* of $V$, denoted by $V^*$. Its elements are sometimes called *covectors*, to make a distinction from the vectors of $V$. Thus, the collection of all forces (covectors) that may act upon the virtual displacements issuing from the configuration $q \in T_q\mathcal{Q}$ is the dual space $T_q^*\mathcal{Q}$, called the *cotangent space of* $\mathcal{Q}$ *at* $q$.