CAMBRIDGE TRACTS IN MATHEMATICS

183 Period Domains over Finite and *p*-adic Fields

Period Domains over Finite and p-adic Fields

JEAN-FRANÇOIS DAT
Université de Paris VI

SASCHA ORLIK
Universität Wuppertal

MICHAEL RAPOPORT
Universität Bonn

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521197694

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 978-0-521-19769-4 Hardback

Contents

Preface

This monograph is a systematic treatise on *period domains* over finite and over p-adic fields. The theory we present here has developed over the past fifteen years. Part of it has already appeared in various research articles or announcements, sometimes without detailed proofs. Our goal here is to present the theory as a whole and to provide complete proofs of the basics of the theory, so that these research articles can be accessed more easily. As it turned out, when working out the details, we had to change the very foundations of the theory quite a bit in some places, especially to accomodate isocrystals over non-algebraically closed fields, and also isocrystals with G-structure. Our hope is that our book can serve as the basis of future research in this exciting area.

Period domains over p-adic fields arose historically at the confluence of two theories: on the one hand, of Fontaine's theory [80] of the "mysterious functor" conjectured by Grothendieck, which relates p-adic Galois representations of p-adic local fields and filtered isocrystals; on the other hand, of the theory of formal moduli spaces of p-divisible groups and their associated period maps [183]. Via the latter theory, they are naturally related to local Langlands correspondences between ℓ-adic representations of the Galois groups of p-adic fields and smooth representations of p-adic Lie groups. In recent times, it became apparent [34, 176] that, via the former theory, period domains also show up in connection with the conjectural p-adic Langlands program relating p-adic representations of the Galois groups of p-adic fields, and p-adic representations of p-adic Lie groups.

There are at least three possible motivations for investigating the period domains of the title. First of all, they are somehow a natural analogue of the Griffiths period domains in Hodge theory. This is true in two respects. Conceptually, a Griffiths period domain is a moduli space for Hodge structures of a certain type; similarly, a p-adic period domain is a moduli space for "weakly admissible filtered isocrystals," which are the p-adic analogues of Hodge structures in Fontaine's theory. Technically, p-adic period domains are defined by a semi-stability formalism; the same is true for Griffiths period domains, although their historical definition is different.

The second motivation is of an arithmetic nature and follows from the relations between Galois representations and p-adic Hodge structures. More precisely, the universal p-adic Hodge structure above a p-adic period domain should conjecturally provide us with a "universal" relative crystalline Galois representation (of a certain type). However, making precise this hope is a very difficult problem of current interest, which lies beyond the scope of this monograph (see the last chapter).

The third motivation was our guide in this monograph. It comes from the formalism used in this theory and its analogies with other topics in algebraic geometry. As already mentioned, the "weakly admissible filtered isocrystals" may be seen as semi-stable objects in the category of filtered isocrystals, once the latter is endowed with a suitable "slope function." Hence there is a direct analogy with the category of vector bundles on a Riemann surface, when endowed with the usual slope function. This analogy has been very fruitful, since many of the "classical" concepts of the theory of vector bundles, such as the Harder–Narasimhan filtration or the GIT criteria for semi-stability, turn out to have natural analogues in the context of filtered isocrystals. Even the theory of G-bundles, i.e., torsors over a reductive group G, has an analogue in the context of filtered isocrystals. This comes from the fact that the category of isocrystals is tannakian, and that the semi-stability condition is compatible with tensor products. In fact, we could have replaced the category of isocrystals by any tannakian category. Choosing the easiest one, namely that of vector spaces over a field, we get rid of the p-adic nature of isocrystals and obtain a theory over any abstract field. It turns out that when the field is finite, there is a moduli space for "semi-stable filtered vector spaces" of a fixed type, and such moduli spaces are the "period domains over finite fields" of the title. Although they are no longer related to any type of Hodge theory, their study remains a good way of approaching that of their p-adic brothers, avoiding most of the intricate technicalities of the p-adic case. For example, when studying period domains over finite fields, only basic algebraic geometry is needed, while in the p-adic case, rigid-analytic geometry is required. Therefore, we like to consider the first two parts of this book as a pedagogical introduction to the formalism of slopes, semi-stability, and related concepts, in the most elementary context where it appears. These first two parts should be accessible to any graduate student with a basic background in algebraic geometry and algebraic group theory.

In the Introduction following this Preface, we give more details on the background of this book and also give a brief description of its contents. Through-out the text, there are numerous examples (which one may also regard as exercises). At the end of each section there are remarks on open questions, on history, and directions to the literature.

We are happy to acknowledge the important contributions of G. Faltings, J.-M. Fontaine, U. Hartl, R. Kottwitz, B. Totaro and Th. Zink to the theory presented here. We also thank M. Cabanes, U. Görtz, M. Harris, X. He, R. Huber, L. Lafforgue, G. Laumon, P. Schneider and A. Strohmaier for helpful discussions. We gave courses on this material at the universities of Wuppertal, Bonn, Leipzig and Paris. We are grateful for the feedback from the audiences of these courses.

Introduction

In this Introduction we give a brief description of the background of this monograph, then explain its scope, and finally give an overview of its contents.

Background 1: Classical Hodge theory The concept of a period domain was created by Griffiths in his work on periods of integrals on algebraic varieties over the field of complex numbers [97]. Let us first explain Griffiths' construction, cf. [96].

Let $H_{\mathbb{R}}$ be a finite-dimensional $\mathbb{R}$-vector space and let $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$. We also fix an integer n and a collection of non-negative integers $\{h^{pq}\}_{p,q}$ which satisfy $h^{pq} = h^{qp}$ and $h^{pq} \neq 0$ only if $p+q = n$, and such that $\sum h^{pq} = \dim H_{\mathbb{R}}$. There is a natural structure of a complex manifold on the set of all Hodge structures of weight n on $H_{\mathbb{R}}$, with h^{pq} as its Hodge numbers. This comes about as follows.

Let $\mathcal{F}$ be the set of all decreasing filtrations of subspaces

$$\cdots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^{p} \subset \mathcal{F}^{p-1} \subset \cdots \subset H_{\mathbb{C}} ,$$

such that $\dim \mathcal{F}^p = \sum_{i \geq p} h^{i,n-i}$. Then $\mathcal{F}$ forms in the obvious way a partial flag variety, and as such has the structure of a smooth projective algebraic variety. The group $\mathrm{GL}(H_{\mathbb{C}})$ acts algebraically and transitively on $\mathcal{F}$. Consider the subset $\mathcal{F}^{\circ}$ of filtrations $\mathcal{F}^{\bullet}$ which satisfy

$$H_{\mathbb{C}} = \mathcal{F}^{p} \oplus \overline{\mathcal{F}^{n-p+1}} \ , \ \text{for every } p \, .$$

Then $\mathcal{F}^{\circ}$ is an open subset of $\mathcal{F}$, and is the parameter space of Hodge structures on $H_{\mathbb{R}}$ of the given type. Indeed, a Hodge structure

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q} \ , \ \overline{H^{p,q}} = H^{q,p} \qquad (0.1)$$

defines the filtration $\mathcal{F}^{\bullet}$ in $\mathcal{F}^{\circ}$ given by

$$\mathcal{F}^{p} = \bigoplus_{i \geq p} H^{i,n-i} \, .$$

Cambridge University Press
978-0-521-19769-4 - Period Domains over Finite and p-adic Fields
Jean-Francois Dat, Sascha Orlik and Michael Rapoport
Frontmatter
More information

Conversely, if $\mathcal{F}^\bullet$ is a point of $\mathcal{F}^\circ$, then $\mathcal{F}^\bullet$ corresponds to the Hodge structure (0.1) with

$$H^{pq} = \mathcal{F}^p \cap \overline{\mathcal{F}^q}\ .$$

For technical reasons, one pays special attention to *polarized* Hodge structures. Let Ψ be a non-degenerate bilinear form on $H_{\mathbb{R}}$ which is symmetric when n is even, and skew-symmetric when n is odd. Let $\mathcal{F}_\Psi$ be the subset of all those filtrations $\mathcal{F}^\bullet$ in $\mathcal{F}$ which satisfy

$$\Psi(\mathcal{F}^p, \mathcal{F}^{n-p+1}) = 0\ ,\ \text{ for every } p\ .$$

Then $\mathcal{F}_\Psi$ is a closed subvariety of $\mathcal{F}$. Let

$$G = \mathrm{Aut}(H_{\mathbb{R}}, \Psi)$$

be the automorphism group of the form Ψ, i.e., either the orthogonal or the symplectic group. The complex Lie group $G(\mathbb{C})$ acts transitively on $\mathcal{F}_\Psi$. In particular, $\mathcal{F}_\Psi$ is a smooth projective variety. Consider the subset $\mathcal{F}_\Psi^\circ$ of filtrations $\mathcal{F}^\bullet$ in $\mathcal{F}^\circ \cap \mathcal{F}_\Psi$ which satisfy

$$\Psi(Cv, \overline{v}) > 0 \text{ for } v \in H_{\mathbb{C}}\ ,\ v \neq 0\ .$$

Here $C = C_{\mathcal{F}^\bullet}$ is the Weil operator $C : H_{\mathbb{C}} \longrightarrow H_{\mathbb{C}}$ defined in terms of the Hodge structure corresponding to $\mathcal{F}^\bullet$ by

$$Cv = i^{p-q}v\ ,\ v \in H^{p,q}\ .$$

Then $\mathcal{F}_\Psi^\circ$ parametrizes all Hodge structures on $H_{\mathbb{R}}$ which are polarized by Ψ and have the h^{pq} as Hodge numbers. The subset $\mathcal{F}_\Psi^\circ$ is open in $\mathcal{F}_\Psi$, and is acted on transitively by $G(\mathbb{R})$. Fixing a base point in $\mathcal{F}_\Psi^\circ$ we therefore have identifications

$$\begin{array}{ccc} \mathcal{F}_\Psi^\circ & \subset & \mathcal{F}_\Psi \\ \Big\| & & \Big\| \\ G(\mathbb{R})/V & \subset & G(\mathbb{C})/P(\mathbb{C})\ , \end{array}$$

where P is a parabolic subgroup of $G_{\mathbb{C}}$, and where the subgroup $V = P(\mathbb{C}) \cap G(\mathbb{R})$ turns out to be compact.

It is $\mathcal{F}_\Psi^\circ$ that is the prototype of a classical period domain, i.e., a period domain in the sense of Griffiths. The name arises from the connection with families of Hodge structures defined by families of algebraic varieties. Let $f : X \longrightarrow S$ be a polarized smooth family of projective algebraic varieties parametrized by a complex variety S. Then for each n, the nth *primitive* cohomology groups of the fibers of f form a local system $PR^n f_*(\mathbb{R})$. Over the universal covering $\tilde{S}$ of S, this local system can be trivialized. Choosing such a trivialization, and associating to a point $\tilde{s} \in \tilde{S}$ with image $s \in S$ the Hodge structure on $PH^n(X_s, \mathbb{R})$, we obtain the *period morphism*

$$\varphi : \tilde{S} \longrightarrow \mathcal{F}^{\circ}. \tag{0.2}$$

Here $\mathcal{F}^{\circ} = \mathcal{F}^{\circ}_{\Psi}$ is the period space relative to the choice of the polarization, of n, of the appropriate Hodge numbers $h^{p,q}$, and of the trivialization of $PR^n f_* \mathbb{R}$ over $\tilde{S}$.

Griffiths and Schmid made a deep study of the differential-geometric properties of period domains, cf. [97]. For any $\mathcal{F}^{\bullet} \in \mathcal{F}^{\circ}$, the Lie algebra $\mathfrak{g}$ of G inherits a real Hodge structure of weight 0 from $\mathrm{End}(H_{\mathbb{R}})$, and the Lie algebra $\mathfrak{p}$ of the parabolic stabilizer P of $\mathcal{F}^{\bullet}$ is the 0th step of the associated Hodge filtration. The holomorphic tangent space of $\mathcal{F}^{\circ}$ at the point corresponding to $\mathcal{F}^{\bullet}$ is naturally isomorphic to $\mathfrak{g}/\mathfrak{p}$. The subspace $\mathfrak{p} \oplus \mathfrak{g}^{-1,1}/\mathfrak{p}$ of $\mathfrak{g}/\mathfrak{p}$ is the fiber at $\mathcal{F}^{\bullet}$ of a $G(\mathbb{C})$-invariant holomorphic subbundle T_h of the holomorphic tangent bundle T, the *horizontal tangent subbundle* [194]. It is a fundamental fact that this subbundle has negative holomorphic sectional curvature bounded away from zero, for a suitable $G(\mathbb{C})$-invariant hermitian form on T. This circumstance allows the application of a version of the Schwartz Lemma. The relevance of this result comes from the fact that any period map φ as in (0.2) is *horizontal*, i.e., $d\varphi(T_{\tilde{S}}) \subset T_h$. These facts have important implications for the local systems $PR^n f_* \mathbb{R}$ defined by families of algebraic varieties (e.g. the proof of Borel [52] of the monodromy theorem).

Also, Schmid [195] studied the L^2-cohomology of the restriction of homogeneous line bundles to period domains, and identified the representations of $G(\mathbb{R})$ afforded by them. This gives a cohomological realization of discrete series representations of $G(\mathbb{R})$.

A striking special case of Griffiths' construction arises from real Hodge structures of type $\{(0, -1), (-1, 0)\}$. These can be identified with complex structures on $H_{\mathbb{R}}$. Choosing a polarization Ψ, the corresponding period domain $\mathcal{F}^{\circ}_{\Psi}$ can be identified with the union of the upper and the lower Siegel halfspaces. In this case, taking for S the moduli space of polarized abelian varieties of type Ψ, the map φ in (0.2) is an isomorphism. In general, the map φ is not even a local isomorphism, since the image of $d\varphi$ lies in the horizontal tangent subbundle.

One can formalize Griffiths' construction following Deligne [52]. Recall that a Hodge structure on $H_{\mathbb{R}}$ corresponds to a homomorphism $h : \mathbb{S} \longrightarrow \mathrm{GL}(H_{\mathbb{R}})$, where

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$$

is the Weil restriction of the multiplicative group. Let G be a connected reductive group over $\mathbb{R}$. A homomorphism $h : \mathbb{S} \longrightarrow G$ is called *polarizable* if it factors through a maximal torus in G which is compact modulo center. Then the period domain associated to the conjugacy class of a polarizable homomorphism $h : \mathbb{S} \longrightarrow G$ is the set $\mathcal{F}(G, h)^\circ$ of conjugates under $G(\mathbb{R})$ of h. Let h_1 be the composite homomorphism

$$(\mathbb{G}_m)_\mathbb{C} \xrightarrow{(1,\mathrm{id})} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_\mathbb{C} = (\mathbb{G}_m)_\mathbb{C} \times (\mathbb{G}_m)_\mathbb{C} \longrightarrow G_\mathbb{C} \; .$$

Any conjugate of h_1 under $G(\mathbb{C})$ defines a filtration on the category of representations of G. The set $\mathcal{F}(G, h)$ of these filtrations can be identified with a generalized flag variety of $G_\mathbb{C}$. We obtain an open embedding

$$\mathcal{F}(G, h)^\circ \hookrightarrow \mathcal{F}(G, h) \, ,$$

and for suitable choices of (G, h) one obtains the examples discussed previously.

In Griffiths' presentation of the theory, the monodromy group acting on $\mathcal{F}(G, h)^\circ$ plays an important role, and the desire to form the quotient of $\mathcal{F}(G, h)^\circ$ by its action is a major reason for considering polarized Hodge structures. Here we will suppress this aspect of the theory, since there is so far no p-adic analogue of these quotients.

Background 2: p-adic Hodge structures The p-adic analogue of a Hodge structure is a *weakly admissible filtered isocrystal*, as defined by Fontaine [80]. Let L be a perfect field of characteristic $p > 0$, and let $K_0 = \mathrm{Quot}(W(L))$ be the fraction field of its ring of Witt vectors. We denote by σ the automorphism of K_0 induced by the Frobenius automorphism of L. An *isocrystal over* L is a finite-dimensional K_0-vector space V, equipped with a bijective σ-linear map $\Phi : V \longrightarrow V$. Let K be a finite field extension of K_0. A *filtered isocrystal* $(V, \Phi, \mathcal{F}^\bullet)$ *over* K is an isocrystal (V, Φ) over L equipped with a (decreasing, exhaustive and separating) $\mathbb{Z}$-filtration $\mathcal{F}^\bullet$ of the K-vector space $V \otimes_{K_0} K$. A filtered isocrystal over K is called *weakly admissible* if

$$\sum\nolimits_x x \dim \mathrm{gr}^x_{\mathcal{F}}(V' \otimes_{K_0} K) \leq \mathrm{ord} \det(\Phi \mid V')$$

for any sub-isocrystal V' of V, with equality for $V' = V$.

Such a structure arises for example from an abelian variety over K with good reduction. In this case (V, Φ) is the rational Dieudonné module of its special fiber, and the filtration $\mathcal{F}^\bullet$ is given by the Hodge filtration of its generic fiber (via the comparison isomorphism with the DeRham cohomology) and has only two jumps, at $x = 0$ and at $x = 1$. Something similar is true of a p-divisible group over K with good reduction. More generally, a filtered isocrystal arises from the ith cohomology group of a smooth projective variety over K with good reduction. In this case, (V, Φ) is given by the ith crystalline cohomology group of the special fiber and again $\mathcal{F}^\bullet$ is given as the Hodge filtration of its generic fiber, which may have more than two jumps.

Note that in the previous example the extension K/K_0 is totally ramified. According to Colmez and Fontaine [44], if the finite extension K/K_0 is totally ramified, the category of weakly admissible filtered isocrystals over K is equivalent to the category of *crystalline p-adic Galois representations of* $\mathrm{Gal}(\bar{K}/K)$ under the *mysterious functor* conjectured by Grothendieck and constructed by Fontaine [80]. In the example of an abelian variety over K with good reduction, this Galois representation is given by the rational p-adic Tate module of its generic fiber, as proved by Breuil [32].

The p-adic analogue of a period domain arises by fixing the isocrystal (V, Φ) and by varying the filtration $\mathcal{F}^\bullet$. More precisely, we fix a function $g : \mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0}$ with $\sum_x g(x) = \dim V$, and consider the partial flag variety $\mathcal{F} = \mathcal{F}(V, g)$ of $\mathbb{Z}$-filtrations $\mathcal{F}^\bullet$ of type g, i.e., such that

$$\dim \mathrm{gr}^x_{\mathcal{F}}(V) = g(x)\ , \qquad \forall x \in \mathbb{Z}\ .$$

Then $\mathcal{F}$ is a smooth projective variety over $\mathbb{Q}_p$, with a transitive action of $\mathrm{GL}(V)$. The locus inside $\mathcal{F}$ corresponding to those filtrations $\mathcal{F}^\bullet$ such that $(V, \Phi, \mathcal{F}^\bullet)$ is weakly admissible is a subset $\mathcal{F}^{\mathrm{wa}}$ of $\mathcal{F} \otimes_{\mathbb{Q}_p} K_0$, which is an admissible open in the sense of rigid-analytic geometry [183]. More precisely, $\mathcal{F}^{\mathrm{wa}}$ is the complement of a p-adic family of Zariski-closed subvarieties of $\mathcal{F} \otimes_{\mathbb{Q}_p} K_0$.

More generally, and imitating Deligne's formalization of the classical period domains, one may perform this construction starting with any triple (G, b, μ). Here G is a connected reductive group over $\mathbb{Q}_p$, and b is an element of $G(K_0)$ and $\mu : (\mathbb{G}_m)_{\bar{\mathbb{Q}}_p} \longrightarrow G_{\bar{\mathbb{Q}}_p}$ is a one-parameter subgroup defined over an algebraic closure $\bar{K}_0$ of K_0. Then μ and any conjugate of μ defines a filtration on the category of representations of G and hence defines a partial flag variety $\mathcal{F}(G, \mu)$ defined over a finite extension E of $\mathbb{Q}_p$ contained in $\bar{K}_0$, the *local Shimura field* associated to (G, μ). Let $\breve{K}_0 = E.K_0$. Then the period domain $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{wa}}$ is an admissible open rigid-analytic subset of $\mathcal{F}(G, \mu) \otimes_E \breve{K}_0$. Its points parametrize weakly admissible triples (G, b, μ'), where (G, b) is fixed and μ' varies in $\mathcal{F}(G, \mu)$. According to [82], if G is quasi-split and L algebraically closed, we have $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{wa}} \neq \emptyset$ if and only if there is an inequality $\nu \leq \mu$ between the *Newton vector* ν of the isocrystal (V, Φ) and the *Hodge vector* of μ.

The best-known example of a p-adic period domain is the Drinfeld halfspace [71]. Let V_0 be a $\mathbb{Q}_p$-vector space of dimension n, and consider the trivial isocrystal $(V, \Phi) = (V_0 \otimes_{\mathbb{Q}_p} K_0, \mathrm{id}_{V_0} \otimes \sigma)$. Let $g : \mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0}$ be the function with

$$g(n-1) = 1, \quad g(-1) = n-1, \quad g(x) = 0 \ \text{ for } x \neq n-1, -1.$$

Cambridge University Press
978-0-521-19769-4 - Period Domains over Finite and p-adic Fields
Jean-Francois Dat, Sascha Orlik and Michael Rapoport
Frontmatter
More information

In this case $\mathcal{F}(V, g) = \mathbb{P}(V) \simeq \mathbb{P}^{n-1}$ is the projective space of lines in V and $\mathcal{F}(V, g)^{\mathrm{wa}}$ is the space $\Omega(V) = \Omega^n$ of all lines not contained in any $\mathbb{Q}_p$-rational hyperplane in V. Drinfeld [71] proved that Ω^n is the generic fiber of an adic formal scheme over Spf $W(L)$ which is the parameter space of certain p-divisible groups of a specific type (*special formal O_D-modules*).

In [183], this is generalized to other families of p-divisible groups, of (EL)- or of (PEL)-type. In this more general case, one obtains a formal scheme $\mathcal{M}$ over Spf $O_{\breve{K}_0}$ (no more adic in general) and a period morphism from the generic fiber of $\mathcal{M}$ to a period domain. This period morphism is an étale rigid-analytic morphism, but in contrast to the Drinfeld case, it is no longer an isomorphism. In the general case, it is conjectured that there is an open subset $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{a}}$ (analytic in the sense of Berkovich) of the analytic space associated to $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{wa}}$, the *admissible subset*, and a local system of p-adic vector spaces over $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{a}}$ such that the fiber in each point $\mathcal{F}^{\bullet}$ corresponds as a p-adic Galois representation with G-structure under the Fontaine functor to the filtered isocrystal with G-structure $(G, b, \mathcal{F}^{\bullet})$ (this more precise version of the conjecture in [183] is due to Hartl [108]).

Let

$$J(\mathbb{Q}_p) = \{g \in G(K_0) \mid gb\sigma(g)^{-1} = b\}$$

be the automorphism group of the isocrystal with G-structure (G, b). The group $J(\mathbb{Q}_p)$ is the group of $\mathbb{Q}_p$-rational points of an algebraic group over $\mathbb{Q}_p$. It acts by naturality on $\mathcal{F}(G, \mu) \otimes_E \breve{K}_0$ preserving the period space $\breve{\mathcal{F}}(G, b, \mu)^{\mathrm{wa}}$. The action of $J(\mathbb{Q}_p)$ also preserves $\mathcal{F}(G, b, \mu)^a$ and this action is lifted to the conjectural local system mentioned above. Imposing level structures on the local system, one obtains a projective system of rigid spaces mapping by surjective étale morphisms to $\breve{\mathcal{F}}(G, b, \mu)^a$. The group $J(\mathbb{Q}_p)$ acts on each member of this projective system, and the group $G(\mathbb{Q}_p)$ acts as Hecke correspondences on the projective system as a whole. From a suitable version of ℓ-adic cohomology in the rigid context, one deduces a $\bar{\mathbb{Q}}_\ell$-vector space with a triple action of $G(\mathbb{Q}_p)$, and $J(\mathbb{Q}_p)$, and the Weil subgroup of Gal$(\bar{E}/E)$. There is a conjecture by Kottwitz [185], generalizing conjectures of Carayol [37] pertaining to the Drinfeld case and the Lubin–Tate case, which describes the precise kind of Langlands correspondence between the representations of these three groups that this triple representation induces (on the *discrete Langlands parameters*; for more general parameters, cf. Harris [106]). In the Drinfeld and the Lubin–Tate cases, in which the respective period domains are Ω^n, resp. $\mathbb{P}^{n-1}$, and in which case the projective system is known to exist, these conjectures have been completely proved very recently, thanks to the results of Boyer, Dat, Faltings, Fargues, and Harris and Taylor.

Cambridge University Press
978-0-521-19769-4 - Period Domains over Finite and p-adic Fields
Jean-Francois Dat, Sascha Orlik and Michael Rapoport
Frontmatter
More information

Background 3: Semi-stability The basic motivation for the present monograph is Faltings' observation [73] that weak admissibility can be viewed as a semi-stability condition. Let $(V, \Phi, \mathcal{F}^\bullet)$ be a filtered isocrystal over K. For any $\mathbb{R}$-filtration $\mathcal{F}^\bullet$ of $V \otimes_{K_0} K$, let

$$\deg_{\mathcal{F}}(V) = \sum\nolimits_x x \dim \mathrm{gr}^x_{\mathcal{F}}(V \otimes_{K_0} K)\ .$$

Let $\mathcal{G}^\bullet$ be the slope filtration of V, i.e., the $\mathbb{Q}$-filtration

$$\mathcal{G}^x = \sum_{-\lambda \geq x} V_{-\lambda}\ ,$$

where $V = \bigoplus V_\lambda$ is the slope decomposition of the isocrystal (V, Φ). Its degree is given by $\deg_{\mathcal{G}}(V) = -\operatorname{ord}\det(\Phi|V)$. Then $(V, \Phi, \mathcal{F}^\bullet)$ is called *semi-stable* if

$$\frac{1}{\dim V'}\big(\deg_{\mathcal{F}}(V') + \deg_{\mathcal{G}}(V')\big) \leq \frac{1}{\dim V}\big(\deg_{\mathcal{F}}(V) + \deg_{\mathcal{G}}(V)\big) \qquad (0.3)$$

for any sub-isocrystal V' of V. Hence $(V, \Phi, \mathcal{F}^\bullet)$ is weakly admissible iff it is semi-stable and if in addition the RHS of (0.3) is equal to zero.

The semi-stability condition is much more flexible than the weak admissibility condition. In particular, it lends itself to analogues in pure linear algebra and hence also to period domains in this pure linear algebra context. More precisely, let V be a finite-dimensional vector space over a field k. Let $\mathcal{F}^\bullet$ be an $\mathbb{R}$-filtration on $V \otimes_k K$, where K is a field extension of k. Then $(V, \mathcal{F}^\bullet)$ is called *semi-stable* if

$$\frac{1}{\dim V'} \deg_{\mathcal{F}'}(V') \leq \frac{1}{\dim V} \deg_{\mathcal{F}}(V)$$

for all k-subspaces V' of V. Here on the LHS, the $\mathbb{R}$-filtration $\mathcal{F}'$ on $V' \otimes_k K$ is the one induced by $\mathcal{F}^\bullet$, and the fractions on both sides are called the *slopes* of V, resp. V'.

We explicitly note that if $L = \mathbb{F}_p$, then a filtered isocrystal becomes an object of linear algebra. In particular, if Φ is the identity automorphism of V, then a filtered isocrystal over K is nothing other than an object $(V, \mathcal{F}^\bullet)$ as above (relative to $k = \mathbb{Q}_p$). It is remarkable that the first kind of object mentioned in Background 1 above can also be phrased in these terms. In fact, by Pink [177], a Hodge structure of weight n on the $\mathbb{R}$-vector space $H_{\mathbb{R}}$ is the same as a semi-stable $\mathbb{Z}$-filtration $\mathcal{F}^\bullet$ on $H_{\mathbb{C}}$ of slope $n/2$.

The filtrations on $V \otimes_k K$ of a fixed type are parametrized by the K-valued points of a partial flag variety $\mathcal{F}$ over k. If k is a finite field, there is a Zariski-open subset $\mathcal{F}^{ss}$ of $\mathcal{F}$ with K-valued points equal to the set of semi-stable pairs $(V, \mathcal{F}^\bullet)$ as above. If k is a non-archimedean local field, the set $\mathcal{F}^{ss}$ is an admissible open rigid-analytic subset of $\mathcal{F}$. These are the prototypes of the period domains in the title. In this context, the theory is stripped of its arithmetic content, and becomes geometric and more elementary. In this monograph we study the geometry of period domains in this context, and address the question of determining their cohomology. We regard the theory developed here as somewhat preliminary to the deep arithmetic questions outlined above.

The scope of this monograph In this monograph we study period domains in the context of semi-stability, in its variants of linear algebra, as well as of the isocrystal variants. Our main purpose is to bring out the analogy between period domains and the moduli spaces of vector bundles on Riemann surfaces [181]. We are especially interested in the geometry of period domains, in particular in determining their cohomology and other topological invariants. Among the topics treated we mention the following.

- The tensor product theorem of Faltings and Totaro, which states that the tensor product of two semi-stable pairs $(V, \mathcal{F}^\bullet)$ and $(V', \mathcal{F}'^\bullet)$ is again semi-stable (in the isocrystals context this was conjectured by Fontaine).
- The machinery of the Harder–Narasimhan filtration, which presents a pair $(V, \mathcal{F}^\bullet)$ as a successive extension of semi-stable pairs in a definite way.
- The relation of the concept of semi-stability of $(V, \mathcal{F}^\bullet)$ to the semi-stability concept in Geometric Invariant Theory (this is due in a special case to van der Put and Voskuil, and in general to Totaro, confirming a conjecture of Rapoport and Zink).
- The structure of the Harder–Narasimhan stratification of the partial flag variety, which reveals an interesting recursive structure of the boundary of the period domain in terms of period domains of smaller dimension.
- The ℓ-adic cohomology with compact supports of period domains (in the case of the Drinfeld halfspace Ω^n, this is due to Drinfeld for $n = 2$ and to Schneider and Stuhler, and to Dat, for arbitrary n, whereas the cohomology *complex* is due to Dat; the case of a general period domain is due to Orlik, and the determination of the Euler–Poincaré characteristic, to Kottwitz and Rapoport).
- The relation between period domains over the field $\mathbb{F}_1$ with one element and thin Bruhat cells and the fibers of the moment map. Here, for the variant of the semi-stability notion over $\mathbb{F}_1$, instead of testing all k-rational subspaces of V, one tests all coordinate subspaces of V with respect to a fixed basis of V. In this variant k can be an arbitrary field.
- The generalization of the theory from $\mathrm{GL}(V)$ to arbitrary reductive groups.
- A systematic treatment of period domains including the case where L is perfect, but not necessarily algebraically closed.

The structure of this monograph The monograph consists of four parts. In the first part (Chapters I–III) we present the theory in its most elementary form. We prove the tensor product theorem in its various variants, and develop the Harder–Narasimhan machinery. We introduce period domains for GL_n over a finite field and over $\mathbb{F}_1$, i.e., in those cases that lead to Zariski-open subsets of generalized flag varieties. We study their stratifications by the Harder–Narasimhan types, resp. by their Harder–Narasimhan polygons. Also, we address the question of determining the cohomology of period domains in this context.

In the second part (Chapters V–VII) the theory in the first part is generalized to the case of an arbitrary reductive group G instead of GL_n. Again, the period domains encountered here are Zariski-open subsets of generalized flag varieties (associated to G). This part is preceded by an interlude on the Tannaka formalism in the context of algebraic groups (Chapter IV).

In the third part (Chapters VIII–X) we pass to the case of a p-adic local field as a base field. In this case we obtain period domains which are admissible rigid-analytic open subsets of generalized flag varieties. They parametrize semi-stable filtered isocrystals. The theory is analogous to that in the first two parts, but is considerably more difficult. In this sense, the first two parts of the monograph may be considered as toy models for the objects in the third part. On the other hand, the theory developed in this last part can be viewed to some degree as a warm-up for the study of the arithmetically significant covering spaces mentioned above.

In a final part (Chapter XI) we give some complements, and review some of the recent work in the area.

Content We now give a chapter-by-chapter description of the contents of the book.

Chapter I is basic for the whole book. In Section 1 we give the general concepts on which the theory rests. In Section 2 we prove the tensor product theorem of Faltings and Totaro; our proof is essentially Totaro's. In Section 3 we introduce the Harder–Narasimhan filtration; we follow Faltings in numbering its filtration steps by the slopes.

Chapter II introduces the period domains attached to a vector space and a dominant co-weight. The basic definition is given in Section 1. In Section 2 we characterize period domains through the Hilbert–Mumford inequality from Geometric Invariant Theory. Section 3 explains the stratification of the generalized flag variety according to Harder–Narasimhan types, resp. Harder–Narasimhan vectors. The subtle difference between these two stratifications – one being a refinement of the other – is a novel phenomenon that does not occur for the space of vector bundles on a Riemann surface. In Section 4 we analyze period domains "over $\mathbb{F}_1$," and connect our theory to the theory of Gelfand–Goresky–MacPherson–Serganova of thin Schubert cells and the moment map.

Chapter III addresses the problem of determining the ℓ-adic cohomology of period domains. Section 1 is devoted to an exposition of the Langlands Lemma from the theory of Eisenstein series, in which we follow closely Laumon [149] and Labesse [141]. In Section 2 we analyze the representations of $\mathrm{GL}_n(\mathbb{F}_q)$ which contain a fixed vector under the Borel subgroup. In particular, we prove by reduction to the representation theory of the symmetric group that the induced representations i_P^G and the generalized Steinberg representations v_P^G form a basis of their Grothendieck group, as P varies over the associate classes of parabolic subgroups. We give the change of basis matrix between these two bases in terms of a complex which is similar to, but in general different from, the Solomon–Tits complex. This is based on a remarkable distributivity property of the representations i_P^G, due to Cabanes. This also leads to the existence of a basis of the group algebra of the Weyl group compatible with the subspaces i_I analogous to i_P^G (however, we strongly believe that it is essentially impossible to give this basis explicitly!). In Section 3 we explain the recursion relation for the Euler–Poincaré characteristic of a period domain using stratification by Harder–Narasimhan types, and we resolve the recursion relation by using the Langlands Lemma. We also explain how, by expressing the result in terms of generalized Steinberg representations, one is led to a formula which gives the cohomology of period domains degree by degree (i.e., not merely the Euler–Poincaré characteristic), and deduce from this a precise vanishing theorem in ℓ-adic cohomology.

Chapter IV gives a brief exposition of some facts from the tannakian formalism that we will need later. We concentrate on the theory of filtrations of fiber functors, and show in particular how a filtration of the natural fiber functor on the representation category of a reductive group can be transferred naturally to any parabolic subgroup.

Chapter V transposes the theory of Chapter I, §1 to general reductive groups. There are two approaches, one *externally* through the tannakian formalism, and one *internally* through group theory – and the main point here is to show that both approaches give the same result. This is done in Sections 1 and 2. Section 3 then transfers the Harder–Narasimhan filtration to this context. We note that the external approach uses the Mumford conjecture from Geometric Invariant Theory (=Haboush's Theorem).

Chapter VI is the analogue of Chapter II for general reductive groups over finite fields. Section 1 defines period domains in this context. Section 2 relates this definition to the definition in terms of Geometric Invariant Theory. This is based on the concept of an invariant inner product on a reductive group due to Totaro, and the proof here is a simplification of Totaro's original proof. Section

3 analyzes the Harder–Narasimhan stratification, and in particular the closure relation among the strata. This is based on an analysis of the structure of the partially ordered uniquely divisible monoid of conjugacy classes of $\mathbb{Q}$-1-PS of G.

Chapter VII starts by introducing the induced representations i_P^G and generalized Steinberg representations v_P^G for general reductive groups over finite fields. It turns out that these generate in general a proper subgroup of the natural Grothendieck group, but that as in the case of GL_n, a basis of this subgroup is given by the i_P^G and also by v_P^G, as P ranges over the associate classes of parabolic subgroups. The proof of this fact is based on the observation that any finite group of Lie type has elliptic regular semi-simple elements (Lusztig informed us that this was known to him previously, but our proof here seems to be the first published one). Section 2 then is the analogue of Chapter III, §3.

Chapter VIII starts with recalling the theory of isocrystals, in particular stressing the special nature of "split semi-simple isocrystals." Then the concepts of semi-stability and of weak admissibility for isocrystals are introduced. Period domains in this new context are defined in Section 2. There are two major differences in our exposition as compared with [183]. First of all, we chose to present the theory in the context of Berkovich spaces instead of rigid-analytic varieties. This is dictated to us through our cohomology calculation in Chapter X, but also facilitates the comparison with the *admissible* set mentioned above. Second, we deal with isocrystals over arbitrary perfect fields. We also give in this section a criterion for when a period domain is non-empty (this question is vacuous in the finite field case). Section 3 introduces in this context the Harder–Narasimhan stratification; in the most general case it is an open problem to determine the set of non-empty strata. Also, the relation between the stratifications by Harder–Narasimhan type and by Harder–Narasimhan vector is difficult outside the split semi-simple case. Section 4 describes period domains for isocrystals in terms of Geometric Invariant Theory, in the split semi-simple case. Section 5 relates isocrystals with an action by a finite extension of $\mathbb{Q}_p$ to Kottwitz's σ-F-spaces, elaborating on a remark in [135].

Chapter IX is technically the most demanding. In Section 1 we introduce the concept of an isocrystal with structure in an arbitrary Tannaka category over $\mathbb{Q}_p$. The best-known example is the case when this Tannaka category is the representation category of an algebraic group over $\mathbb{Q}_p$. However, more general tannakian categories are in fact needed in order to deal with the Harder–Narasimhan formalism, even if one starts with a G-isocrystal in the usual sense. In order to deal effectively with this concept, we introduce the notion of an augmented group scheme over the Tannaka category of isocrystals over L. In Section 2 we use this formalism to transpose semi-stability and Harder–Narasimhan filtration to this context. In Section 3 we analyze the automorphism group of an isocrystal with structure in a tannakian category. In Section 4 we generalize to this context the analysis of the space of conjugacy classes of $\mathbb{Q}$-1-PS from Chapter VI, §3. In Section 5 we introduce period domains. A technically important remark is that we may change an augmented group scheme by a "weak isomorphism" without changing the period domain. In Section 6 we analyze the Harder–Narasimhan stratification in this context. The situation in general is quite complicated, but it simplifies in the split semi-simple case. The final Section 7 is devoted to the operation of restriction of scalars in this context.

Chapter X is devoted to the cohomology of period domains attached to *basic* isocrystals. In Section 1 we discuss generalized Steinberg representations in the p-adic case. The situation here is different from the finite field case, and in fact well-known and much simpler. Section 2 then treats the cohomology of period domains. Here we use basic facts on the ℓ-adic cohomology of Berkovich spaces, which we treat in an axiomatic way. By changing an augmented group scheme within its weak isomorphism class, we are reduced to a situation which is essentially identical to that over a finite field. Hence the same proof as in Chapter VII, §2 yields a result that is very similar to the one in the linear algebra context.

Chapter XI has a more informal character. In Section 1 we discuss the "fundamental complex," the main ingredient of the determination of the cohomology of period domains (and not merely their Euler–Poincaré characteristic). The geometry behind this complex also enters into Section 2, where we compare period domains over a finite field to the other class of algebraic varieties attached to finite groups of Lie type, the Deligne–Lusztig varieties. In particular, we compare properties like affineness and simple connectivity for both classes of varieties. In Section 3 we discuss some special features of the Drinfeld space. As already pointed out in [186], this period domain is quite atypical of period domains in general, but the comparison is useful to keep in mind. Section 3 concerns the conjectural local system of $\mathbb{Q}_p$-vector spaces on an open subset of a period domain, and discusses the results of Faltings [75] and Hartl [109] concerning them in the weight 1 case. In Section 4 we discuss the results of Dat [48] concerning the cohomology *complex* of the Drinfeld space. At least the *splitting theorem* holds for general period domains, but whether one can extend the main results of [48] to general period domains is an unresolved question.