# THE MATHEMATICAL LANGUAGE OF QUANTUM THEORY

For almost every student of physics, their first course on quantum theory raises puzzling questions and creates an uncertain picture of the quantum world. This book presents a clear and detailed exposition of the fundamental concepts of quantum theory: states, effects, observables, channels and instruments. It introduces several up-to-date topics, such as state discrimination, quantum tomography, measurement disturbance and entanglement distillation. A separate chapter is devoted to quantum entanglement. The theory is illustrated with numerous examples, reflecting recent developments in the field. The treatment emphasizes quantum information, though its general approach makes it a useful resource for graduate students and researchers in all subfields of quantum theory. Focusing on mathematically precise formulations, the book summarizes the relevant mathematics.

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Vladimir Buzek, editor of the journal.

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## From Uncertainty to Entanglement

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# Preface

Quantum theory is not an easy subject to master. Trained in the everyday world of macroscopic objects like locomotives, elephants and watermelons, we are insensitive to the beauty of the quantum world. Many quantum phenomena are revealed only in carefully planned experiments in a sophisticated laboratory. Some features of quantum theory may seem contradictory and inconceivable in the framework set by our experience. Rescue comes from the language of mathematics. Its mighty power extends the limits of our apprehension and gives us tools to reason systematically even if our practical knowledge fails. Mastering the relevant mathematical language helps us to avoid unnecessary quantum controversies.

Quantum theory, as we understand it in this book, is a general framework. It is not so much about *what is out there*, but, rather, determines constraints on *what is possible* and *what is impossible*. This type of constraint is familiar from the theory of relativity and from thermodynamics. We will see that quantum theory is also a framework, and one of great interest, where these kinds of question can be studied.

What are the main lessons that quantum theory has taught us? The answer, of course, depends on who you ask. Two general themes in this book reflect our answer: uncertainty and entanglement.

*Uncertainty*. Quantum theory is a statistical theory and there seems to be no way to escape its probabilistic nature. The intrinsic randomness of quantum events is the seed of this uncertainty. There are various different ways in which it is manifested in quantum theory. We will discuss many of these aspects, including the nonunique decomposition of a mixed state into pure states, Gleason's theorem, the no-cloning theorem, the impossibility of discriminating nonorthogonal states and the unavoidable disturbance caused by the quantum measurement process.

*Entanglement*. The phenomenon of entanglement provides composite quantum systems with a very puzzling and counterintuitive flavour. Many of its consequences dramatically contradict our classical experience. Measurement outcomes observed by different observers can be entangled in a curious way, allowing for the violation of local realism or the teleportation of quantum states. It is fair to say that entanglement is the key resource for quantum information-processing tasks.

It should be noted that both these themes, uncertainty and entanglement, have puzzled quantum theorists since the beginning of the quantum age. Uncertainty can be traced back to Werner Heisenberg, while the word 'entanglement' was coined by Erwin Schrödinger. After many decades uncertainty and entanglement are still under active research, probably more so than ever before.

The evolution of this book had several stages, quite similar to the life cycle of a frog. Its birth dates back to a series of lectures the authors gave in 2007–8 in the Research Center for Quantum Information, Bratislava. The audience consisted of Ph.D. students working on various subfields of quantum theory. The principal aim of the lectures was to introduce a common language for the core part of quantum theory and to formulate some fundamental theorems in a mathematically precise way.

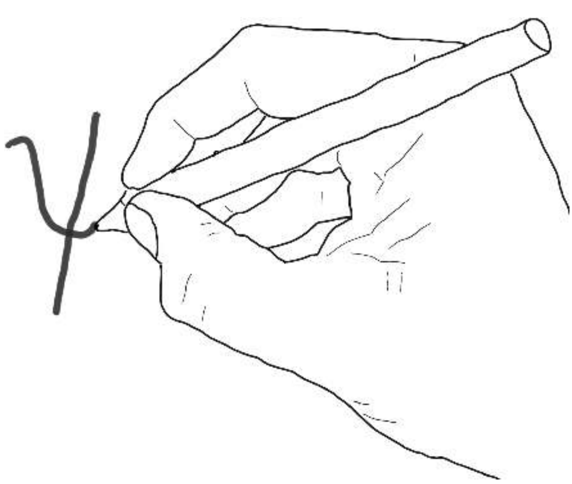
The positive response from the students encouraged us to hatch the egg, and the tadpole stage of this book was an article based on the lecture notes. The article appeared in *Acta Physica Slovaca* in August 2008. The metamorphosis from a tadpole to an adult frog took more than two years. During that stage we benefited greatly from the comments of our friends and colleagues. We hope that this grown-up frog can serve as a guide for students wishing to get an overview of the mathematical formulation of quantum theory.

*Acknowledgement*. This book would have never come into existence without the guidance, encouragement and help of Vladimír Bužek. Many friends and colleagues read parts of earlier versions of the manuscript and gave us valuable comments and support. Our special thanks are addressed to Paul Busch, Andreas Doering, Viktor Eisler, Sergej Nikolajevič Filippov, Stan Gudder, Jukka Kiukas, Pekka Lahti, Leon Loveridge, Daniel McNulty, Harri Mäkelä, Marco Piani, Daniel Reitzner, Tomáš Rybár, Michal Sedlák, Peter Staňo and Kari Ylinen. Teiko Heinosaari is grateful to the Alfred Kordelin Foundation for financial support.

# Introduction

*What is this book about*? This book is an *introduction* to the basic structure of quantum theory. Our aim is to present the most essential concepts and their properties in a pedagogical and simple way but also to keep a certain level of mathematical precision. On top of that our intention is to illustrate the formalism in examples that are closely related to current research problems. As a result, the book has a quantum information flavor although it is not a quantum information textbook. The ideas related to quantum information are presented as consequences or applications of the basic quantum formalism.

Many textbooks on quantum physics concentrate either on finite- or infinite-dimensional Hilbert spaces. In this book the idea has been to treat finite- and infinite-dimensional Hilbert-space formalisms on the same footing. To keep the book at a reasonable size and so as not to be drawn into mathematical technicalities, we have sometimes compromised and presented a theorem in a general form



but have given its proof only in the finite-dimensional case. The last chapter, on the mathematical aspects of the phenomenon of entanglement, is written entirely in a finite-dimensional setting.

*What this book is not about*. We should perhaps warn the reader that this book is not about the different interpretations of quantum theory: we mostly avoid discussions of the philosophical consequences of the theory. These issues are important and interesting, but a proper understanding is easier to achieve if one knows the mathematical structure reasonably well first. Naturally, one cannot grasp the theory without some minimal interpretation. We have tried to keep the discussion away from controversial questions that do not (yet) have commonly agreed frameworks. Actually, even though two quantum theorists might disagree on some deeper foundational issues, they would agree on the basic predictions of quantum theory, such as measurement outcome probabilities. The reader will notice that many topics that could (and perhaps should) be included under the title *The Mathematical Language of Quantum Theory* are missing. For instance, the book *Mathematical Concepts of Quantum Mechanics* by Gustafson and Sigal has almost the same title as the present book but an almost disjoint content (actually, it can be recommended as a complement to this book). We believe that our book provides a compact and coherent overview of one branch of the mathematical language of quantum theory.

*Prerequisites*. We assume that the reader has already met quantum theory and has perhaps studied an elementary course on quantum mechanics. We also assume a basic knowledge of real analysis and linear algebra. We imagine that a typical reader is a Master's or Ph.D. student who wants to broaden his or her knowledge and learn a framework into which quantum concepts fit naturally. Clear definitions of basic concepts and their properties, together with a reasonably comprehensive index, should make the book useful as a reference work also.

*Structure of the book*. The book consists of 28 sections, grouped into six thematic chapters. Each section is divided into several subsections, which typically treat one question or topic. In the first chapter we recall the basics of Hilbert spaces. In each of the following five chapters we concentrate on one or two key concepts of quantum formalism. Within the body of the text there are short exercises where the reader is asked to verify a formula. These exercises are easy and straightforward, and hints are often given.