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Hilbert space refresher

Quantum theory, in its conventional formulation, is built on the theory of Hilbert spaces and operators. In this chapter we go through this basic material, which is central for the rest of the book. Our treatment is mainly intended as a refresher and a summary of useful results. It is assumed that the reader is already familiar with some of these concepts and elementary results, at least in the case of finite-dimensional inner product spaces.

We present proofs for propositions and theorems only if the proof itself is considered to be instructive and illustrative. This gives us the freedom to present the material in a topical order rather than in the strict order of mathematical implication. Good references for this chapter are the functional analysis textbooks by Conway [45], Pedersen [113] and Reed and Simon [121]. These books also contain the proofs that we skip here.

1.1 Hilbert spaces

As an introduction, before a formal definition is given, one may think of a Hilbert space as the closest possible generalization of the inner product spaces $\mathbb{C}^d$ to infinite dimensions. Actually, there are no finite-dimensional Hilbert spaces (up to isomorphisms) other than $\mathbb{C}^d$ spaces. The crucial requirement of *completeness* becomes relevant only in infinite-dimensional spaces. This defining property of Hilbert spaces guarantees that they are well-behaved mathematical objects, and many calculations can be done almost as easily as in $\mathbb{C}^d$.

1.1.1 Finite- and infinite-dimensional Hilbert spaces

Let $\mathcal{H}$ be a complex vector space. We recall that a complex-valued function $\langle\cdot|\cdot\rangle$ on $\mathcal{H} \times \mathcal{H}$ is an *inner product* if it satisfies the following three conditions for all $\varphi, \psi, \phi \in \mathcal{H}$ and $c \in \mathbb{C}$:

- $\langle \varphi | c\psi + \phi \rangle = c \langle \varphi | \psi \rangle + \langle \varphi | \phi \rangle$ (linearity in the second argument),
- $\overline{\langle \varphi | \psi \rangle} = \langle \psi | \varphi \rangle$ (conjugate symmetry),
- $\langle \psi | \psi \rangle > 0 \quad \text{if} \quad \psi \neq 0$ (positive definiteness).

A complex vector space $\mathcal{H}$ with an inner product defined on it is an *inner product space*. An alternative name for an inner product is a *scalar product*, and then naturally one speaks of scalar product spaces.

The defining conditions for an inner product have some elementary consequences. First notice that linearity and conjugate symmetry imply that

$$\langle c\varphi + \phi | \psi \rangle = \bar{c} \langle \varphi | \psi \rangle + \langle \phi | \psi \rangle$$

and that $\langle 0 | \psi \rangle = \langle \psi | 0 \rangle = 0$.

An often-used implication, which follows from the previous equation and positive definiteness, is that

$$\langle \psi | \psi \rangle = 0 \qquad \Rightarrow \qquad \psi = 0 \,. \tag{1.1}$$

An elementary but important result for inner product spaces is the *Cauchy–Schwarz inequality*: if $\varphi, \psi \in \mathcal{H}$ then

$$|\langle \varphi | \psi \rangle|^2 \leq \langle \varphi | \varphi \rangle \langle \psi | \psi \rangle \ . \tag{1.2}$$

Moreover, equality occurs if and only if φ and ψ are linearly dependent, meaning that $\varphi = c\psi$ for some complex number c. The Cauchy–Schwarz inequality will be used constantly in our calculations.

A word of warning: unlike in quantum mechanics textbooks, in most functional analysis textbooks inner products are linear in the *first* argument. This is, of course, just a harmless difference in convention, and one can define

$$\langle \varphi | \psi \rangle_{\text{quantum book}} = \langle \psi | \varphi \rangle_{\text{maths book}}$$

to obtain an inner product that is linear in the second argument.

Example 1.1 (*Inner product space* $\mathbb{C}^d$)
Let $\mathbb{C}^d$ denote the vector space of all d-tuples of complex numbers. For two vectors $\varphi = (\alpha_1, \ldots, \alpha_d)$ and $\psi = (\beta_1, \ldots, \beta_d)$, the inner product $\langle \varphi | \psi \rangle$ is defined as follows:

$$\langle \varphi | \psi \rangle = \sum_{j=1}^{d} \bar{\alpha}_j \beta_j \ . \tag{1.3}$$

There are also other inner products on $\mathbb{C}^d$ (try to invent one!), but when referring to $\mathbb{C}^d$ we will always assume that the inner product is that defined in (1.3). △

An *isomorphism* is, generally speaking, a structure-preserving bijective mapping. Inner product spaces have a vector space structure and the additional structure given by the inner product. Hence, in the context of inner product spaces an isomorphism is defined as follows.

Definition 1.2 Two inner product spaces $\mathcal{H}$ and $\mathcal{H}'$ are *isomorphic* if there is a bijective linear mapping $U : \mathcal{H} \to \mathcal{H}'$ such that

$$\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle \tag{1.4}$$

for all $\varphi, \psi \in \mathcal{H}$. The mapping U is an *isomorphism*.

Two isomorphic spaces are completely indistinguishable as abstract inner product spaces. Isomorphisms are like costumes; two Hilbert spaces may be intrinsically the same but in the morning they have chosen different costumes and therefore they look different.

An essential concept regarding inner product spaces is orthogonality. Two vectors $\varphi, \psi \in \mathcal{H}$ are called *orthogonal* if $\langle \varphi | \psi \rangle = 0$, in which case we write $\varphi \perp \psi$. A set of vectors $X \subset \mathcal{H}$ is called *orthogonal* if any two distinct vectors belonging to X are orthogonal. Among other things, the orthogonality property can be used to define the dimension of an inner product space. First, we will draw a distinction between finite and infinite dimensions.

Definition 1.3 Let $\mathcal{H}$ be an inner product space. If for any positive integer d there exists an orthogonal set of d vectors, then $\mathcal{H}$ is *infinite dimensional*. Otherwise $\mathcal{H}$ is *finite dimensional*.

We recall the following characterization of finite-dimensional inner product spaces. This basic result is usually proved in linear algebra courses.

Proposition 1.4 If $\mathcal{H}$ is a finite-dimensional inner product space then there is a positive integer d such that:

- there are d nonzero orthogonal vectors;
- for $d' > d$, any set of d' nonzero vectors contains nonorthogonal vectors.

The number d is called the *dimension* of $\mathcal{H}$. A finite-dimensional inner product space of dimension d is isomorphic to $\mathbb{C}^d$.

Exercise 1.5 For finite-dimensional inner product spaces, the dimension can be defined equivalently as the maximal number of nonzero linearly independent vectors. Let $X = \{\varphi_1, \ldots, \varphi_n\}$ be an orthogonal set of nonzero vectors in a d-dimensional Hilbert space $\mathcal{H}$. Show that X is a linearly independent set.

[Hint: Start from the equation $c_1\varphi_1 + \cdots + c_n\varphi_n = 0$. You need to show that $c_1 = \cdots = c_n = 0$.]

Not all inner product spaces are finite dimensional. The following example, which we also use later, demonstrates this fact.

Example 1.6 (*Inner product space* $\ell^2(\mathbb{N})$)
We denote by $\mathbb{N}$ the set of natural numbers, including 0. Let $\ell^2(\mathbb{N})$ be a set of functions $f : \mathbb{N} \to \mathbb{C}$ such that the sum $\sum_{j=0}^{\infty} |f(j)|^2$ is finite. The formula

$$\langle f|g\rangle = \sum_{j=0}^{\infty} \overline{f(j)}g(j) \tag{1.5}$$

defines an inner product on $\ell^2(\mathbb{N})$. (The only nonobvious part in showing that the formula (1.5) is an inner product is verifying that the sum $\sum_{j=0}^{\infty} \overline{f(j)}g(j)$ converges whenever $f, g \in \ell^2(\mathbb{N})$. This follows from Hölder's inequality.) For each $k \in \mathbb{N}$, let δ_k be the *Kronecker function*, defined as

$$\delta_k(j) = \begin{cases} 1 & \text{if } j = k\,, \\ 0 & \text{if } j \neq k\,. \end{cases}$$

It follows that

$$\langle \delta_k|\delta_\ell\rangle = 0$$

whenever $k \neq \ell$. The inner product space $\ell^2(\mathbb{N})$ is infinite dimensional since the collection of Kronecker functions is an orthogonal set. △

Every inner product space $\mathcal{H}$ is a *normed space*, the norm being defined as

$$\|\psi\| \equiv \sqrt{\langle\psi|\psi\rangle}\,. \tag{1.6}$$

The fact that the real-valued function $\psi \mapsto \|\psi\|$ defined in (1.6) is a norm means that the following three conditions are satisfied for all $\varphi, \psi \in \mathcal{H}$ and $c \in \mathbb{C}$:

- $\|\varphi\| \geq 0$ and $\|\varphi\| = 0$ if and only if $\varphi = 0$;
- $\|c\varphi\| = |c|\,\|\varphi\|$;
- $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$ (triangle inequality).

The first two properties follow immediately from the defining conditions of inner products. The third is easy to prove using the Cauchy–Schwarz inequality (1.2):

$$\begin{aligned} \|\varphi + \psi\|^2 &= \langle\varphi|\varphi\rangle + \langle\varphi|\psi\rangle + \langle\psi|\varphi\rangle + \langle\psi|\psi\rangle \\ &\leq \langle\varphi|\varphi\rangle + 2|\langle\varphi|\psi\rangle| + \langle\psi|\psi\rangle \\ &\leq \langle\varphi|\varphi\rangle + 2\sqrt{\langle\varphi|\varphi\rangle}\sqrt{\langle\psi|\psi\rangle} + \langle\psi|\psi\rangle \\ &= \left(\|\varphi\| + \|\psi\|\right)^2. \end{aligned}$$

Three useful formulas are stated in the following exercises.

Exercise 1.7 (*Pythagorean formula*)
Let φ and ψ be orthogonal vectors in an inner product space $\mathcal{H}$. Prove that the following equality, known as the *Pythagorean formula*, holds:

$$\|\varphi + \psi\|^2 = \|\varphi\|^2 + \|\psi\|^2 \ . \tag{1.7}$$

[Hint: Use (1.6) and expand the left-hand side of the equation.] In contrast with the Pythagorean formula for the real inner product space $\mathbb{R}^d$, (1.7) can hold even if φ and ψ are not orthogonal. Find two vectors which demonstrate this fact. [Hint: It is essential that $\mathcal{H}$ is a *complex* inner product space.]

Exercise 1.8 (*Bessel's inequality*)
Let $\{\psi_1, \ldots, \psi_n\}$ be an orthogonal set of n unit vectors (i.e. $\|\psi_j\| = 1$) in an inner product space $\mathcal{H}$. Show that if $\varphi \in \mathcal{H}$ then

$$\sum_{j=1}^{n} |\langle\psi_j|\varphi\rangle|^2 \leq \|\varphi\|^2 \ . \tag{1.8}$$

[Hint: Start from the fact that

$$0 \leq \left\langle \varphi - \sum_{j=1}^{n} \langle\psi_j|\varphi\rangle\, \psi_j \,\middle|\, \varphi - \sum_{j=1}^{n} \langle\psi_j|\varphi\rangle\, \psi_j \right\rangle$$

and expand the right-hand side of this inequality.]

Exercise 1.9 (*Parallelogram law*)
Let φ and ψ be vectors in an inner product space $\mathcal{H}$. Prove that the following equality, known as the *parallelogram law*, holds:

$$\|\varphi + \psi\|^2 + \|\varphi - \psi\|^2 = 2\,\|\varphi\|^2 + 2\,\|\psi\|^2 \ . \tag{1.9}$$

[Hint: Use (1.6) and expand the left-hand side of the equation.] It is of interest to note that the converse is also true: a normed linear space is an inner product space if the norm satisfies the parallelogram law. (A proof of this fact can be found e.g. in [62], Theorem 6.1.5.)

The norm induces a *metric* on $\mathcal{H}$. The distance between two vectors ψ and φ is given by

$$d(\psi, \varphi) \equiv \|\psi - \varphi\| \ . \tag{1.10}$$

With the concept of distance defined, it makes sense to speak about metric concepts in an inner product space $\mathcal{H}$. For instance, we note that, for each vector ψ, the mapping $\varphi \mapsto \langle\psi|\varphi\rangle$ from $\mathcal{H}$ to $\mathbb{C}$ is *continuous*. (This is a direct consequence of the Cauchy–Schwarz inequality.)

A metric space is called *complete* if every Cauchy sequence is convergent. (A sequence $\{\varphi_j\}$ is a Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer N_ε such that $d(\varphi_j, \varphi_k) < \varepsilon$ whenever $j, k > N_\varepsilon$.) Loosely speaking, completeness means that every sequence which looks convergent is indeed convergent.

Since inner product spaces are not only metric spaces but also normed spaces, there is an alternative characterization of completeness. Namely, a normed space is complete if and only if every absolutely convergent series is convergent. (A series $\sum_{j=1}^{\infty} \phi_j$ is called absolutely convergent if $\sum_{j=1}^{\infty} \|\phi_j\| < \infty$.)

Every finite-dimensional inner product space is automatically complete. For infinite-dimensional inner product spaces this is not true. Furthermore there is a special name for inner product spaces that are complete.

Definition 1.10 A complete inner product space is called a *Hilbert space*.

Completeness has several useful consequences, which make Hilbert spaces much easier to deal with than general inner product spaces. One consequence is the existence of basis expansions (subsection 1.1.2).

An orthogonal set $X \subset \mathcal{H}$ is an *orthonormal set* if each vector $\psi \in X$ has unit norm. An *orthonormal basis* for a Hilbert space $\mathcal{H}$ is a maximal orthonormal set; this means that there is no other orthonormal set containing it as a proper subset. A useful criterion for the maximality of an orthonormal set $X \subset \mathcal{H}$ is the following: if ψ is orthogonal to all vectors in X then $\psi = 0$.

It can be shown that every Hilbert space has an orthonormal basis and, moreover, that all orthonormal bases of a given Hilbert space have the same cardinality. A Hilbert space $\mathcal{H}$ is called *separable* if it has a countable orthonormal basis.

Example 1.11 ($\ell^2(\mathbb{N})$ *continued*)
It can be shown that the inner product space $\ell^2(\mathbb{N})$ is complete. The set of Kronecker functions $\{\delta_0, \delta_1, \ldots\}$ is an orthonormal basis for $\ell^2(\mathbb{N})$. This can be seen by using the criterion for maximality mentioned earlier. If $f \in \ell^2(\mathbb{N})$ satisfies $\langle \delta_k | f \rangle = 0$ then $f(k) = 0$. Hence, a function f which is orthogonal to all Kronecker functions is identically zero. We conclude that $\ell^2(\mathbb{N})$ is a separable infinite-dimensional Hilbert space. △

The following proposition should be compared with Proposition 1.4.

Proposition 1.12 Any separable infinite-dimensional Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.

The idea behind Proposition 1.12 is simple, and we will give an outline of the proof without going into the details. Fix an orthonormal basis $\{\varphi_k\}_{k=0}^{\infty}$ for a separable Hilbert space $\mathcal{H}$. For each vector $\psi \in \mathcal{H}$, define a function $\tilde{\psi} : \mathbb{N} \to \mathbb{C}$ by $\tilde{\psi}(j) = \langle \varphi_j | \psi \rangle$. It then follows that $\tilde{\psi} \in \ell^2(\mathbb{N})$ and the correspondence $\psi \mapsto \tilde{\psi}$ is an isomorphism between $\mathcal{H}$ and $\ell^2(\mathbb{N})$.

From now on, all Hilbert spaces are assumed to be separable.

In other words, all our Hilbert spaces are either finite dimensional or countably infinite dimensional. Typically, we denote Hilbert spaces by the letters $\mathcal{H}$ or $\mathcal{K}$. Sometimes we use $\mathcal{H}_d$ for a finite d-dimensional Hilbert space. To avoid trivial statements, we will also assume that the dimension d of our Hilbert space is at least 2.

1.1.2 Basis expansion

Let $\mathcal{H}$ be either a finite-dimensional or separable infinite-dimensional Hilbert space and let $\{\varphi_k\}_{k=1}^d$ be an orthonormal basis for $\mathcal{H}$. We recall that this means the following three things:

- $\langle \varphi_k | \varphi_k \rangle = 1$ for every k;
- $\langle \varphi_j | \varphi_k \rangle = 0$ for every $j \neq k$;
- if $\langle \psi | \varphi_k \rangle = 0$ for every k then $\psi = 0$.

In the case of a finite-dimensional Hilbert space, it is often useful to understand an orthonormal basis as a list of vectors rather than just as a set. In other words, we order the elements of the orthonormal basis. There is then a unique correspondence between the vectors and the d-tuples of complex numbers. This is actually just another way to express the isomorphism statement at the end of Proposition 1.4. Similarly, in the case of a separable infinite-dimensional Hilbert space we take an orthonormal basis to mean a sequence of orthogonal vectors (rather than just a set) whenever this is convenient.

Once an orthonormal basis is fixed, we can write every vector $\psi \in \mathcal{H}$ in terms of a *basis expansion*:

$$\psi = \sum_{k=1}^{d} \langle \varphi_k | \psi \rangle \, \varphi_k \, . \tag{1.11}$$

The exact meaning of this formula depends on whether the Hilbert space is finite or infinite dimensional.

In finite-dimensional Hilbert spaces the basis expansion is just a finite sum. The basis expansion simply expresses the fact that each vector ψ can be written as a linear combination of the basis vectors φ_k.

Exercise 1.13 Suppose that $d < \infty$ and $\psi = \sum_{k=1}^{d} c_k \varphi_k$. Prove the following: if $\{\varphi_k\}_{k=1}^{d}$ is an orthonormal basis then $c_k = \langle \varphi_k | \psi \rangle$. [Hint: Start from $\psi = \sum_{k=1}^{d} c_k \varphi_k$ and take the inner product with φ_k on both sides.]

In the case of an infinite-dimensional Hilbert space, the basis expansion (1.11) is a convergent series. To see this, first observe that

$$\sum_{k=1}^{n} \| \langle \varphi_k | \psi \rangle \, \varphi_k \|^2 = \sum_{k=1}^{n} | \langle \varphi_k | \psi \rangle |^2 \leq \| \psi \|^2$$

for any $n = 1, 2, \ldots$, where we have applied Bessel's inequality (1.8). This implies that the series

$$\sum_{k=1}^{\infty} \| \langle \varphi_k | \psi \rangle \, \varphi_k \|^2$$

converges (since the sequence of the partial sums is increasing and bounded). We then recall from subsection 1.1.1 that in a Hilbert space every absolutely convergent series is convergent. Since we have seen that the series is convergent, the basis expansion (1.11) is easy to verify. We note that the vector $\psi - \sum_{k=1}^{\infty} \langle \varphi_k | \psi \rangle \, \varphi_k$ is orthogonal to every basis vector φ_ℓ. But since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis, it follows that $\psi - \sum_{k=0}^{\infty} \langle \varphi_k | \psi \rangle \, \varphi_k = 0$. Therefore, (1.11) holds.

Let us remark that an infinite-dimensional Hilbert space does not contain a countable set of vectors that would give any other vector as a linear combination of some subset of this set. (By a linear combination we always mean a finite sum.) We can conclude from the basis expansion that any vector can be approximated arbitrarily well by a linear combination of basis vectors.

Example 1.14 (*Basis expansion in* $\ell^2(\mathbb{N})$)
We saw earlier that the Hilbert space $\ell^2(\mathbb{N})$ consists of complex functions f on $\mathbb{N}$ such that the sum $\sum_{k=0}^{\infty} |f(k)|^2$ is finite and the Kronecker functions form an orthonormal basis in $\ell^2(\mathbb{N})$. If a function f is nonzero only at a finite number of points k, then clearly we can write it as

$$f = \sum_{k: f(k) \neq 0} f(k) \delta_k \, . \tag{1.12}$$

This is nothing other than the basis expansion of f in the Kronecker basis. The formula (1.12) is true for all $f \in \ell^2(\mathbb{N})$, but in general it is a convergent series and need not be a finite sum. △

The complex numbers $\langle \varphi_k | \psi \rangle$ in (1.11) are called the *Fourier coefficients* of ψ with respect to the orthonormal basis $\{\varphi_k\}_{k=1}^{d}$. They give not only the basis expansion of ψ but also the best approximation of ψ if one is allowed to use only some fixed subset of the basis vectors. Namely, fix a positive integer $m \leq d$. For any choice of m complex numbers $c_1, \ldots, c_m$, we have

$$\left\| \psi - \sum_{k=1}^{m} \langle \varphi_k | \psi \rangle \varphi_k \right\| \leq \left\| \psi - \sum_{k=1}^{m} c_k \varphi_k \right\| . \tag{1.13}$$

To see that this inequality holds, we write the square of the right-hand side in the form

$$\left\| \psi - \sum_{k=1}^{m} c_k \varphi_k \right\|^2 = \| \psi \|^2 - \sum_{k=1}^{m} |\langle \varphi_k | \psi \rangle|^2 + \sum_{k=1}^{m} |c_k - \langle \varphi_k | \psi \rangle|^2$$

and then observe that the last sum is positive unless $c_k = \langle \varphi_k | \psi \rangle$ for every $k = 1, \ldots, m$.

Another useful thing about Fourier coefficients is that one can use them to calculate the norm of a vector. The norm of ψ in (1.11) is given by *Parseval's formula*:

$$\| \psi \|^2 = \sum_{k=1}^{d} |\langle \varphi_k | \psi \rangle|^2 . \tag{1.14}$$

This is obtained from the following chain of equalities:

$$\begin{aligned} \| \psi \|^2 = \langle \psi | \psi \rangle &= \left\langle \sum_{k=1}^{d} \langle \varphi_k | \psi \rangle \varphi_k \middle| \sum_{j=1}^{d} \langle \varphi_j | \psi \rangle \varphi_j \right\rangle \\ &= \sum_{k=1}^{d} \sum_{j=1}^{d} \overline{\langle \varphi_k | \psi \rangle} \langle \varphi_j | \psi \rangle \langle \varphi_k | \varphi_j \rangle = \sum_{k=1}^{d} |\langle \varphi_k | \psi \rangle|^2 . \end{aligned}$$

In the infinite-dimensional case, the fact that the sums can be taken out of the inner product is justified by the continuity of the latter.

1.1.3 Example: $L^2(\Omega)$

Up to now, we have encountered only one particular infinite-dimensional Hilbert space, namely $\ell^2(\mathbb{N})$. At the abstract level, there are no other separable infinite-dimensional Hilbert spaces as they are all isomorphic (recall Proposition 1.12). However, in applications Hilbert spaces typically have some concrete form. The benefit of using one particular concrete form rather than another is that certain operators may be easier to handle or a calculation may be easier to perform.

One very useful class of concrete Hilbert spaces is that consisting of so-called *square integrable functions*. Actually, $\ell^2(\mathbb{N})$ also belongs to this class but then the integration is just a sum on $\mathbb{N}$ and square integrability means that the sum

$\sum_{j=0}^{\infty} |f(j)|^2$ is finite. (In technical terms, the integration is with respect to the counting measure on $\mathbb{N}$.)

To introduce other examples of this type of Hilbert space, let Ω be the real line $\mathbb{R}$ or an interval on $\mathbb{R}$. We denote by $L^2(\Omega)$ the set of complex-valued functions on Ω for which the integral $\int_\Omega |f(x)|^2 dx$ is finite. To be more precise, functions are required to be measurable and two functions are identified if they are equal almost everywhere. With these conventions, $L^2(\Omega)$ is a separable infinite-dimensional Hilbert space equipped with an inner product

$$\langle f|g\rangle = \int_\Omega \overline{f(x)} g(x)\, dx\,. \tag{1.15}$$

There is, of course, some work to do to show that (1.15) is an inner product and that it makes $L^2(\Omega)$ a separable Hilbert space. These details can be found, in for instance, the textbooks by Rudin [122] or Folland [61].

We conclude that a unit vector in $L^2(\Omega)$ is a function $f : \Omega \to \mathbb{C}$ satisfying $\|f\|^2 = \int_\Omega |f(x)|^2 dx = 1$. It is possible (and convenient) to choose an orthonormal basis for $L^2(\Omega)$ consisting of continuous functions. For a nice example, let $\Omega = [0, 2\pi)$. A function f on $[0, 2\pi)$ can be alternatively thought of as a periodic function on $\mathbb{R}$ with period 2π. Clearly, for each $n \in \mathbb{Z}$, the function $e_n(x) := e^{inx}$ belongs to $L^2([0, 2\pi))$. Two functions e_n and e_m with $n \neq m$ are orthogonal, since

$$\langle e_n|e_m\rangle = \int_0^{2\pi} e^{i(m-n)x}\, dx = 0 \tag{1.16}$$

and we have $\|e_n\| = \sqrt{2\pi}$. One can actually prove that the set

$$\{e_n/\sqrt{2\pi} : n \in \mathbb{Z}\}$$

is an orthonormal basis in $L^2([0, 2\pi))$. For any $f \in L^2([0, 2\pi))$, one obtains

$$\langle e_n|f\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)\, dx\,.$$

These numbers are (up to a constant factor) just the usual Fourier coefficients of f.

This example of square integrable functions has a natural extension from complex-valued functions to vector-valued functions. We denote by $L^2(\Omega; \mathbb{C}^d)$ the set of functions from Ω to $\mathbb{C}^d$ for which the integral $\int_\Omega \|f(x)\|^2\, dx$ is finite. The norm inside the integral is the norm in $\mathbb{C}^d$. The inner product in $L^2(\Omega; \mathbb{C}^d)$ is defined as

$$\langle f|g\rangle = \int_\Omega \langle f(x)|g(x)\rangle\, dx\,,$$

where the inner product under the integral sign is the inner product of $\mathbb{C}^d$. In a similar way we can start from any Hilbert space $\mathcal{H}$ and define $L^2(\Omega; \mathcal{H})$.