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Martin Bojowald
Excerpt
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Introduction

Einstein's equation

$$G_{ab} = 8\pi G T_{ab} \tag{1.1}$$

presents a complicated system of non-linear partial differential equations of up to second order for the space-time metric g_{ab}. As a tensorial equation, it determines the structure of space-time in a covariant and coordinate-independent way. Nevertheless, coordinates are often chosen to arrive at specific solutions, and the Einstein tensor is split into its components in the process. In component form, one then notices that some of the equations are of first order only; they do not appear as evolution equations but rather as constraints on the initial values that can be posed for the second-order part of Einstein's equation. Moreover, some components of the metric do not appear as second-order derivatives at all.

Physically, all these properties taken together capture the self-interacting nature of the gravitational field and its intimate relationship with the structure of space-time. Einstein's equation is not to be solved on a given background space-time, its solutions rather determine how space-time itself evolves starting with the structure of an initial spatial manifold. General covariance allows one to express solutions in any coordinate system and to relate solutions based only on different choices of coordinates in consistent ways. Consistency is ensured by properties of the first-order part of the equation, and coordinate redundancy by the different behaviors of metric components. All these properties are thus crucial, but they make the theory rather difficult to analyze and to understand.

Instead of solving Einstein's equation just as one set of coupled partial differential equations, the use of *geometry* provides important additional insights by which much information can be gained in an elegant and systematic way. There is, first, space-time itself which is equipped with a Riemannian structure and thus encodes the gravitational field in a geometrical way. Geometry allows many identifications of observable space-time quantities, and it provides means to understand space-time globally and to arrive at general theorems, for instance regarding singularities. These structures can be analyzed with differential geometry, which is provided in most introductory textbooks on general relativity and will be assumed at least as basic knowledge in this book. (More advanced geometrical topics are provided in the Appendix.) We will be assuming familiarity with the first part of the book by Wald (1984), and use similar notations.

In addition to space-time, also the solution space to Einstein's equation, just like the solution space of any field theory, is equipped with a special kind of geometry: symplectic or Poisson geometry as the basis of canonical methods. General properties of solution spaces regarding gauge freedom, as originally analyzed by Dirac, are best seen in such a setting. In this book, the traditional treatment of systems with constraints following Dirac's classification will be accompanied by a mathematical discussion of geometrical properties of the solution spaces involved. With this combination, a more penetrating view can be developed, showing how natural several of the distinctions made by Dirac are from a mathematical perspective. In gravity, these techniques become especially important for understanding the solutions of Einstein's equation and their relationships to each other and to observables. They provide exactly the systematic tools required to understand the evolution problem and consistency of Einstein's equation and the meaning of the way in which space-time structure is described, but they are certainly not confined to this purpose. Canonical techniques are relevant for many applications, including cosmology of homogeneous models and perturbations around them, and collapse models of matter distributions into black holes. Regarding observational aspects of cosmology, for instance, canonical methods provide systematic tools to derive gauge-invariant observables and their evolution. Finally, canonical methods are important when the theory is to be quantized to obtain quantum gravity.

We will first illustrate the appearance and application of canonical techniques in gravity by the example of isotropic cosmology. What we learn in this context will be applied to general relativity in Chapter 3, in which the main versions of canonical formulations — those due to Arnowitt, Deser and Misner (ADM) (2008) and a reformulation in terms of Ashtekar variables — are derived. At the same time, mathematical techniques of symplectic and Poisson geometry will be developed. Applications at this general level include a discussion of the initial-value problem as well as an exhibition of canonical methods and their results in numerical relativity. Canonical matter systems will also be discussed in this chapter.

Just as one often solves Einstein's equation in a symmetric context, symmetry-reduced models provide interesting applications of the canonical equations. Classes of these models, general issues of symmetry reduction, and perturbations around symmetric models are the topic of Chapter 4. The main cosmological implications of general relativity will be touched upon in the process. From the mathematical side, the general theory of connections and fiber bundles will be developed in this chapter. Spherically symmetric models, then, do not only provide insights about black holes, but also illustrate the symmetry structures behind the canonical formulation of general relativity (in terms of Lie algebroids).

Chapter 5 does not introduce new canonical techniques, but rather, shows how they are interlinked with other, differential geometric methods often used to analyze global properties of solutions of general relativity. These include geodesic congruences, singularity theorems, the structure of horizons, and matching techniques to construct complicated solutions from simpler ones. The class of physical applications in this chapter will mainly be black holes, regarding properties of their horizons as well as models for their formation in gravitational collapse.

Chaper 6 then provides concluding discussions with a brief, non-exhaustive outlook on the application to canonical quantum gravity. This topic would require an entire book for a detailed discussion, and so here we only use the final chapter to provide a self-contained link from the methods developed in the main body of this book to the advanced topic of quantum gravity. Several books exist by now dedicated to the topic of canonical quantum gravity, to which we refer for further studies.

This book grew out of a graduate course on "Advanced Topics in General Relativity" held at Penn State, taking place with the prerequisite of a one-semester introduction to general relativity that normally covers the usual topics up to the Schwarzschild space-time. In addition to extending the understanding of Einstein's equation, this course has the aim to provide the basis for research careers in the diverse direction of gravitational physics, such as numerical relativity, cosmology and quantum gravity. The material contained in this book is much more than could be covered in a single semester, but it has been included to provide a wider perspective and some extra background material. If the book is used for teaching, choices of preferred topics will have to be made. The extra material is sometimes used for independent studies projects, as happened during the preparation of this book.

I am grateful to a large number of colleagues and students for collaborations and explorations over several years, in particular to Rupam Das, Xihao Deng, Golam Hossain, Mikhail Kagan, George Paily, Juan Reyes, Aureliano Skirzewski, Thomas Strobl, Rakesh Tibrewala and Artur Tsobanjan, with whom I have worked on issues related to the material in this book. Finally, I thank Hans Kastrup for having instilled in me a deep respect for Hamiltonian methods. One of the clearest memories from my days as a student is a homework problem of a classical-mechanics class taught by Hans Kastrup. It was about Hamilton–Jacobi methods, epigraphed with the quote "Put off thy shoes from off thy feet, for the place whereon thou standest is holy ground."

2

Isotropic cosmology: a prelude

Cosmology presents the simplest dynamical models of space-time by assuming space to be homogeneous and isotropic on large scales. This reduces the line element to Friedmann–Lemaître–Robertson–Walker (FLRW) form:

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma_k^2 \tag{2.1}$$

with the spatial line element

$$\mathrm{d}\sigma_k^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{2.2}$$

of a 3-space of constant curvature. Only this form is compatible with the assumption of spatial isotropy — the existence of a 6-dimensional isometry group acting transitively on spatial slices $t=$ const and on tangent spaces — as we will derive in detail in Chapter 4.2.1. The only free functions are the *lapse function* $N(t)$ and the *scale factor* $a(t)$, while the constant curvature parameter k can take the values zero (spatial flatness), plus one (positive spatial curvature; 3-sphere) or minus one (negative spatial curvature; hyperbolic space).

Both the lapse function and the scale factor must be non-zero, and can be assumed positive without loss of generality. The lapse function determines the clock-rate by which the coordinate t measures time. It can be absorbed by using cosmological proper time[1] τ defined via $\mathrm{d}\tau = N(t)\mathrm{d}t$, a differential equation for $\tau(t)$. With a positive $N(t)$, $\tau(t) = \int N(t)\mathrm{d}t$ is a monotonic function and can thus be inverted to obtain $t(\tau)$ to be inserted in $a(t)$ in the metric if we want to transform from t to τ.

The scale factor measures the expansion or contraction of space in time. For a spatially flat model, it can be rescaled by a constant which would simply change the spatial coordinates. (For models with non-vanishing spatial curvature, the rescaling freedom of coordinates is conventionally fixed by normalizing k to be ± 1.) However, unlike $N(t)$ it cannot be completely absorbed in coordinates while preserving the isotropic form of the line element. Its relative change such as the *Hubble parameter* $\dot{a}/a$ or relative acceleration parameters thus do have physical meaning. They are subject to the dynamical equations of isotropic cosmological models.

[1] The notion of proper time refers to observers, in the present case to co-moving ones staying at a fixed point in space and passively following the expansion or contraction of the universe.

2.1 Equations of motion

The dynamics of gravity is determined by the Einstein–Hilbert action

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^4x \left(\frac{1}{16\pi G} \sqrt{-\det g}\, R + \mathcal{L}_{\mathrm{matter}} \right) \tag{2.3}$$

where g_{ab} is the space-time metric, $\mathcal{L}_{\mathrm{matter}}$ a Lagrangian density for matter and $R = g^{ab}R_{ab} = g^{ab}R_{acb}{}^{c}$ the Ricci scalar. We will later verify that this action indeed produces Einstein's equation; see Example 3.7.

2.1.1 Reduced Lagrangian

For an isotropic metric (2.1) it is easy to derive the Ricci scalar:

$$R = 6\left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{k}{a^2} \right) . \tag{2.4}$$

With $\det g = -r^4 \sin^2(\vartheta) N(t)^2 a(t)^6/(1-kr^2)$ we then have the *reduced gravitational action*

$$S^{\mathrm{iso}}_{\mathrm{grav}}[a,N] = \frac{3V_0}{8\pi G} \int \mathrm{d}t\, Na^3 \left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{k}{a^2} \right) \tag{2.5}$$

$$= -\frac{3V_0}{8\pi G} \int \mathrm{d}t \left(\frac{a\dot{a}^2}{N} - kaN \right) \tag{2.6}$$

integrating by parts in the second step. Note that we do not need to integrate over all of space (and in fact cannot always do so in a well-defined way if space is non-compact) because the geometry of our isotropic space-time is the same everywhere for constant t. An arbitrary constant $V_0 := \int \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi r^2 \sin\vartheta/\sqrt{1-kr^2}$ thus arises after picking a compact integration region. From now on we will be assuming that V_0 equals one, which can always be achieved by picking a suitable region to integrate over. This identifies the reduced gravitational Lagrangian as

$$L^{\mathrm{iso}}_{\mathrm{grav}} = -\frac{3}{8\pi G} \left(\frac{a\dot{a}^2}{N} - kaN \right) . \tag{2.7}$$

Note that it does not depend on the time derivative of the lapse function.

In this derivation, we are commuting the two steps involved in the derivation of reduced equations of motion: we do not use the full equations of motion that are obtained from varying the action (as done explicitly in Example 3.7) and then insert a special symmetric form of solutions, but insert this symmetric form, (2.1), into the action and then derive equations of motion from variations. There is no guarantee in general that this is in fact allowed: equations of motion correspond to extrema of the action functional; if the action is restricted before variation, some extrema might be missed. The reduced action may, in some cases, not produce the correct equations of motion. In the case of interest here, however, it is true that one can proceed in this way and we do so because it is simpler. We will come back to this problem (called symmetric criticality) from a more general perspective in Chapter 4.2.2.

2.1.2 Canonical analysis

In the reduced action, our free functions of time are $a(t)$ and $N(t)$, which lead to the canonical variables $(a, p_a; N, p_N)$. Momenta are derived in the usual way as

$$p_a = \frac{\partial L^{\rm iso}_{\rm grav}}{\partial \dot{a}} = -\frac{3}{4\pi G}\frac{a\dot{a}}{N}, \qquad p_N = \frac{\partial L^{\rm iso}_{\rm grav}}{\partial \dot{N}} = 0\,. \tag{2.8}$$

Because the Lagrangian does not depend on $\dot{N}$, the momentum p_N vanishes identically and is not a degree of freedom. Its vanishing rather presents a *primary constraint* on the canonical variables and their dynamics. Constraints of this form are associated with gauge freedom of the action, and $p_N = 0$ corresponds to the freedom of redefining time: as seen from the line element, $N(t)$ can be absorbed in the choice of the coordinate t. It thus cannot be a physical degree of freedom, and is not granted a non-trivial momentum.

Proceeding with the canonical analysis, we derive the *gravitational Hamiltonian*

$$H^{\rm iso}_{\rm grav} = \dot{a}p_a + \dot{N}p_N - L^{\rm iso}_{\rm grav} = -\frac{2\pi G}{3}\frac{Np_a^2}{a} - \frac{3}{8\pi G}kaN\,. \tag{2.9}$$

Or, keeping a general matter contribution with Hamiltonian $H_{\rm matter}$ and our primary constraint, which can be added since it vanishes, we have the *total Hamiltonian*

$$H^{\rm iso}_{\rm total} = H^{\rm iso}_{\rm grav} + H^{\rm iso}_{\rm matter} + \lambda p_N \tag{2.10}$$

where $\lambda(t)$ is an arbitrary function. This Hamiltonian determines evolution by Hamiltonian equations of motion

$$\dot{N} = \frac{\partial H^{\rm iso}_{\rm total}}{\partial p_N} = \lambda \tag{2.11}$$

$$\dot{p}_N = -\frac{\partial H^{\rm iso}_{\rm total}}{\partial N} = \frac{2\pi G}{3}\frac{p_a^2}{a} + \frac{3}{8\pi G}ka - \frac{\partial H^{\rm iso}_{\rm matter}}{\partial N} \tag{2.12}$$

$$\dot{a} = \frac{\partial H^{\rm iso}_{\rm total}}{\partial p_a} = -\frac{4\pi G}{3}\frac{Np_a}{a} \tag{2.13}$$

$$\dot{p}_a = -\frac{\partial H^{\rm iso}_{\rm total}}{\partial a} = -\frac{2\pi G}{3}\frac{Np_a^2}{a^2} + \frac{3}{8\pi G}Nk - \frac{\partial H^{\rm iso}_{\rm matter}}{\partial a}\,. \tag{2.14}$$

The first equation, (2.11), tells us again that $N(t)$ is completely arbitary as a function of time, for $\lambda(t)$ remained free when we added the primary constraint to the Hamiltonian. The second equation, (2.12), implies a *secondary constraint* because $p_N = 0$ must be valid at all times, and thus $\dot{p}_N = 0$, or

$$-\frac{2\pi G}{3}\frac{p_a^2}{a} - \frac{3}{8\pi G}ka + \frac{\partial H^{\rm iso}_{\rm matter}}{\partial N} = 0\,. \tag{2.15}$$

The third equation, (2.13), reproduces the definition (2.8) of the momentum p_a, whose equation of motion (2.14) then provides a second-order evolution equation for a.[2]

2.1.3 Scalar field

This set of equations for the gravitational variables is accompanied by equations for matter degrees of freedom, if present, which can be derived analogously from an explicit matter Hamiltonian. In isotropic cosmology, the only matter source compatible with the exact symmetries is a scalar field φ, which in minimally coupled form has an action

$$S_{\mathrm{scalar}}[\varphi] = -\int \mathrm{d}^4x\sqrt{-\det g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right) . \tag{2.16}$$

(More generally, there can be non-minimal coupling terms to gravity of the form $\frac{1}{2}\xi R\varphi^2$ with the Ricci scalar R. Any other curvature couplings would require a parameter of dimension length, which is not available at the classical level; only quantum corrections could provide extra terms making use of the Planck length $\ell_{\mathrm{P}} = \sqrt{G\hbar}$.) For isotropic metrics and spatially homogeneous φ, this reduces to the Lagrangian

$$L^{\mathrm{iso}}_{\mathrm{scalar}} = \frac{a^3}{2N}\dot{\varphi}^2 - Na^3V(\varphi) \tag{2.17}$$

which we now analyze canonically.

The scalar has a momentum

$$p_\varphi = \frac{\partial L^{\mathrm{iso}}_{\mathrm{scalar}}}{\partial\dot{\varphi}} = \frac{a^3\dot{\varphi}}{N} \tag{2.18}$$

and the Hamiltonian is

$$H^{\mathrm{iso}}_{\mathrm{scalar}}(\varphi, p_\varphi) = \dot{\varphi}p_\varphi - L^{\mathrm{iso}}_{\mathrm{scalar}}(\varphi, p_\varphi) = \frac{Np_\varphi^2}{2a^3} + Na^3V(\varphi) . \tag{2.19}$$

Hamiltonian equations of motion are $\dot{\varphi} = \partial H^{\mathrm{iso}}_{\mathrm{scalar}}/\partial p_\varphi = Np_\varphi/a^3$ which reproduces (2.18) and

$$\dot{p}_\varphi = -\frac{\partial H^{\mathrm{iso}}_{\mathrm{scalar}}}{\partial\varphi} = -Na^3V'(\varphi) . \tag{2.20}$$

2.1.4 Friedmann equations

In order to bring the equations in more conventional form, we use (2.13) to eliminate p_a in (2.15) and (2.14). In this way we obtain the *Friedmann equation*

$$\left(\frac{\dot{a}}{aN}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\frac{1}{a^3}\frac{\partial H^{\mathrm{iso}}_{\mathrm{matter}}}{\partial N} \tag{2.21}$$

[2] Had we not chosen to set $V_0 = 1$, the Lagrangian, the momenta, and the Hamiltonian would have remained multiplied with V_0. In all equations of motion, both sides scale in the same way when V_0 is changed; the dynamics is thus independent of the choice of V_0.

and the *Raychaudhuri equation*

$$\frac{(\dot{a}/N)^{\cdot}}{aN} = -\frac{4\pi G}{3}\left(\frac{1}{a^3}\frac{\partial H^{\rm iso}_{\rm matter}}{\partial N} - \frac{1}{Na^2}\frac{\partial H^{\rm iso}_{\rm matter}}{\partial a}\right). \tag{2.22}$$

For the scalar field,[3]

$$\frac{1}{a^3}\frac{\partial H^{\rm iso}_{\rm scalar}}{\partial N} = \frac{p_\varphi^2}{2a^6} + V(\varphi) \tag{2.23}$$

and

$$-\frac{1}{Na^2}\frac{\partial H^{\rm iso}_{\rm scalar}}{\partial a} = 3\left(\frac{p_\varphi^2}{2a^6} - V(\varphi)\right).$$

The first-order Hamiltonian equations of motion for φ and p_φ can be combined to a second-order equation for φ, the Klein–Gordon equation

$$\frac{(\dot{\varphi}/N)^{\cdot}}{N} - 3\frac{\dot{a}}{Na}\frac{\dot{\varphi}}{N} + V'(\varphi) = 0. \tag{2.24}$$

2.2 Matter parameters

In a matter Hamiltonian, formulated in canonical variables, any N-dependence arises only from the measure factor $\sqrt{-\det g}$, and thus the Hamiltonian must be proportional to N. For a homogeneous space-time, we then have

$$\frac{\partial H_{\rm matter}}{\partial N} = \frac{1}{N}H_{\rm matter} = E \tag{2.25}$$

as the matter Hamiltonian measured in proper time, or the energy. (Energy is frame-dependent, in the case of isotropic cosmology amounting to a reference to N. We will exhibit the general frame dependence in the full expressions in Chapter 3.6.) Furthermore, we use the spatial volume $V = a^3$ to define the *energy density*[4]

$$\rho := \frac{E}{V} = \frac{H_{\rm matter}}{Na^3} \tag{2.26}$$

and *pressure*

$$P := -\frac{\partial E}{\partial V} = -\frac{1}{3Na^2}\frac{\partial H_{\rm matter}}{\partial a}. \tag{2.27}$$

These quantities, unlike E, are independent under rescaling a or changing the time coordinate. (In an isotropic universe, these two quantities completely determine the stress-energy tensor

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \tag{2.28}$$

[3] All partial derivatives require the other *canonical* variables to be held fixed while taking them since these are the independent variables in Hamiltonian equations of motion. Thus, $\partial p_\varphi/\partial a = 0$ even though p_φ, according to (2.18), appears to depend on a. However, $\partial\dot{\varphi}/\partial a \neq 0$ because $\dot{\varphi}$ is not a canonical variable held fixed for $\partial/\partial a$.

[4] With our choice of $V_0 = 1$, this is the energy in our integration region divided by the volume of the region. Thanks to homogeneity, this ratio must be the energy density everywhere.

in perfect-fluid form, such that $\rho = T_{ab}u^a u^b$ and $P = T_{ab}v^a v^a$ where $u^a = (\partial/\partial\tau)^a$ with $u_a u^a = -1$ is the fluid 4-velocity and v^a is a unit spatial vector satisfying $v^a u_a = 0$ and $v_a v^a = 1$.)

Thus, we rewrite the Friedmann and Raychaudhuri equations (2.21) and (2.22) as

$$\left(\frac{\dot{a}}{aN}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{2.29}$$

$$\frac{(\dot{a}/N)^{\cdot}}{aN} = -\frac{4\pi G}{3}(\rho + 3P)\,. \tag{2.30}$$

This set of one first- and one second-order differential equation implies, as a consistency condition, the *continuity equation*

$$\frac{\dot{\rho}}{N} + 3\frac{\dot{a}}{Na}(\rho + P) = 0\,. \tag{2.31}$$

One can also derive this equation from the conservation equation of a perfect-fluid stress-energy tensor.

Notice that these equations only refer to observable quantities, which are the scaling-independent matter parameters ρ and P as well as the *Hubble parameter*

$$\mathcal{H} = \frac{\dot{a}}{aN} \tag{2.32}$$

and the *deceleration parameter*

$$q = -\frac{a(\dot{a}/N)^{\cdot} N}{\dot{a}^2}\,. \tag{2.33}$$

There is no dependence on the rescaling of the scale factor in these parameters, nor is there a dependence on the choice of time coordinate. In fact, all time derivatives appear in the invariant proper-time form $\mathrm{d}/\mathrm{d}\tau = N^{-1}\mathrm{d}/\mathrm{d}t$.

Example 2.1 (de Sitter expansion)

If pressure equals the negative energy density, $P = -\rho$, the energy density and thus the Hubble parameter $\mathcal{H}$ must be constant in time by virtue of (2.31). This behavior is realized when matter contributions are dominated by a positive cosmological constant Λ. In proper time, we then have the Friedmann equation $\dot{a} = \mathcal{H}a$, solved by $a = a_0 \exp(\mathcal{H}\tau)$.

Next to proper time, a parameter often used is conformal time with $N = a$, making (2.1) with $k = 0$ conformally equivalent to flat space-time. In this example, the transformation to conformal time is obtained as $\eta(\tau) = \int e^{-\mathcal{H}\tau}\mathrm{d}\tau = -(\mathcal{H}a(\tau))^{-1}$. Thus, the scale factor as a function of conformal time behaves as $a(\eta) = -(\mathcal{H}\eta)^{-1}$. While proper time can take the whole range of real values, conformal time must be negative. (None of these coordinates covers all of de Sitter space with a flat spatial slicing.) A finite conformal-time interval approaching $\eta \to 0$ corresponds to an infinite amount of proper time. The divergence of $a(\eta)$ for $\eta \to 0$ is thus only a coordinate effect but with no physical singularity since no observer, who must experience proper time in the rest frame, can live to experience the divergence. For later use we note the relationships $a''/a = 2/\eta^2 = 2\dot{a}^2 = 2\mathcal{H}_{\mathrm{conf}}^2$ for conformal-time *derivatives (denoted by primes) and the* conformal Hubble parameter $\mathcal{H}_{\rm conf} = a'/a = \dot{a} \neq \mathcal{H}$.

In order to solve the equations of isotropic cosmology, an equation of state $P(\rho)$ must be known, or matter degrees of freedom subject to additional equations of motion must be specified. In the preceding example, this was the simple relationship $P = -\rho$. More generally, one may assume a linear relationship $P = w\rho$ with a constant equation-of-state parameter w.

Example 2.2 (Perfect fluid)
A perfect fluid satisfies the equation of state $P = w\rho$ with a constant w. For $w = 0$, the fluid is called dust*, and for $w = 1/3$ we have* radiation *(see Chapter 3.6.3). Solving the continuity equation (2.31) implies that*

$$\rho \propto a^{-3(w+1)} . \tag{2.34}$$

For dust, energy density $\rho \propto a^{-3}$ is thus just being diluted as the universe expands, while radiation with $\rho \propto a^{-4}$ has an additional red-shift factor. In proper time, $N = 1$, and for a spatially flat universe, $k = 0$, the Friedmann equation $(\dot{a}/a)^2 \propto a^{-3(w+1)}$ shows that $a(\tau) \propto (\tau - \tau_0)^{2/(3+3w)}$ for $w \neq -1$ and $a(\tau) \propto \exp(\sqrt{8\pi G\Lambda/3}\,\tau)$ for $w = -1$, where the matter contribution is only from a cosmological constant $\Lambda = \rho = -P$. In conformal time, $N = a$, the Friedmann equation reads $(a'/a^2)^2 \propto a^{-3(w+1)}$ and gives $a(\eta) \propto (\eta - \eta_0)^{2/(1+3w)}$ for $w \neq -1/3$.

In the example, we can see the following properties:

1. Deceleration, $q > 0$, is realized for $w > -\frac{1}{3}$, which includes all normal forms of matter.
2. Solutions are in general *singular*:
   (i) a can diverge at finite proper time for $w < -1$.
   (ii) a can vanish at finite proper time for $w > -1$, which includes in particular dust and radiation.

In both cases, the Ricci scalar diverges and the Friedmann equation ceases to provide a well-posed initial-value problem. (For the limiting value of $w = -1$, we have the maximally symmetric, and thus non-singular, de Sitter space-time of Example 2.1.)

2.3 Energy conditions

In order to distinguish classes of general matter sources, those not necessarily characterized by a single parameter such as w, with physically and causally reasonable properties one defines energy conditions which a stress-energy tensor should satisfy:

Weak energy condition, WEC $T_{ab}v^av^b \geq 0$ must be satisfied for all timelike v^a (which by continuity implies that it is also satisfied for null vector fields). If this is true, the local energy density will be non-negative for any observer.

In an isotropic space-time the stress-energy tensor $T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)$ must be of perfect-fluid form, for which the WEC directly implies that $\rho = T_{ab}u^au^b \geq 0$, and