PART I

Constructions, examples, and structure theory

1

Overview of pseudo-reductivity

1.1 Comparison with the reductive case

The notion of pseudo-reductivity is due to Borel and Tits. We begin by defining this concept, as well as some related notions.

Definition 1.1.1 Let k be a field, G a smooth affine k-group. The *k-unipotent radical* $\mathscr{R}_{u,k}(G)$ (resp. *k-radical* $\mathscr{R}_k(G)$) is the maximal smooth connected unipotent (resp. solvable) normal k-subgroup of G. A *pseudo-reductive k-group* is a smooth connected affine k-group G such that $\mathscr{R}_{u,k}(G) = 1$.

In the definition of pseudo-reductivity, it is equivalent to impose the stronger condition that G contains no nontrivial smooth connected k-subgroups U such that $U_{\overline{k}} \subseteq \mathscr{R}_u(G_{\overline{k}})$. To prove the equivalence it suffices to show that $U \subseteq \mathscr{R}_{u,k}(G)$, or equivalently that U is contained in a smooth connected unipotent *normal* k-subgroup. The smooth connected k_s-subgroup U' in G_{k_s} generated by the $G(k_s)$-conjugates of U_{k_s} is normal in G_{k_s} and satisfies $U'_{\overline{k}} \subseteq \mathscr{R}_u(G_{\overline{k}})$, so U' is unipotent. By construction, U' is $\mathrm{Gal}(k_s/k)$-stable, so it descends to a smooth connected unipotent normal k-subgroup of G that contains U, as desired. A consequence of this argument is that if N is a smooth connected normal k-subgroup of a smooth connected affine k-group G, so $\mathscr{R}_u(N_{\overline{k}}) \subseteq \mathscr{R}_u(G_{\overline{k}})$, then $\mathscr{R}_{u,k}(N) \subseteq \mathscr{R}_{u,k}(G)$. In particular, if G is pseudo-reductive then so is N. In particular, every smooth connected k-subgroup of a commutative pseudo-reductive k-group is pseudo-reductive.

Pseudo-reductive k-groups are called *k-reductive* in Springer's book [Spr]. To define a related notion of pseudo-semisimplicity, triviality of $\mathscr{R}_k(G)$ (assuming connectedness of G) is a necessary but not sufficient condition for the right definition. The reason is that for any imperfect field k, there are smooth connected affine k-groups G such that $\mathscr{R}_k(G) = 1$ and $\mathscr{D}(G) \neq G$ (see Example 11.2.1). We will study pseudo-semisimplicity in §11.2.

Over a perfect field k, pseudo-reductivity coincides with reductivity for smooth connected affine k-groups G because $\mathscr{R}_{u,k}(G)_{\overline{k}} = \mathscr{R}_u(G_{\overline{k}})$ if k is perfect. In contrast, for any imperfect field k there are many pseudo-reductive k-groups G that are not reductive. We will see interesting examples in §1.3, but we wish to begin with an elementary commutative example of a non-reductive pseudo-reductive group to convey a feeling for pseudo-reductivity.

Our example will rest on the Weil restriction functor $\mathrm{R}_{k'/k}$ relative to a finite extension of fields k'/k that is not separable. Weil restriction is a familiar operation in the separable case (where it is analogous to the operation of viewing a complex manifold as a real manifold of twice the dimension), but it is not widely known for general finite extensions k'/k. We will use it extensively with inseparable extensions, so we first review a few basic facts about this functor, referring the reader to §A.5 and §A.7 for a more thorough discussion.

If $B \to B'$ is a finite flat map between noetherian rings (e.g., a finite extension of fields) and if X' is a quasi-projective B'-scheme, then the *Weil restriction* $X = \mathrm{R}_{B'/B}(X')$ is a separated B-scheme of finite type (even quasi-projective) characterized by the functorial property $X(A) = X'(B' \otimes_B A)$ for all B-algebras A. The discussion of Weil restriction in §A.5 treats the general algebro-geometric setting as well as the special case of group schemes. For any smooth connected affine k'-group G', the Weil restriction $G = \mathrm{R}_{k'/k}(G')$ is an affine k-group scheme of finite type characterized by the property

$$G(A) = G'(k' \otimes_k A)$$

functorially in k-algebras A. By Proposition A.5.11, the k-group $G = \mathrm{R}_{k'/k}(G')$ is smooth and connected. Its dimension is $[k' : k] \dim G'$, since $\mathrm{Lie}(G)$ is the Lie algebra over k underlying $\mathrm{Lie}(G')$ (Corollary A.7.6).

Remark 1.1.2 Beware that if k'/k is inseparable then the "pushforward" functor $\mathrm{R}_{k'/k}$ from affine k'-schemes to affine k-schemes has some surprising properties in comparison with the more familiar separable case. This will be illustrated in Examples 1.3.2 and 1.3.5.

Example 1.1.3 Now we give our first example of a non-reductive pseudo-reductive group. Let k be an imperfect field of characteristic $p > 0$, and let k'/k be a purely inseparable finite extension of degree $p^n > 1$. Consider the smooth k-group $G = \mathrm{R}_{k'/k}(\mathrm{GL}_1)$ of dimension p^n. (Loosely speaking, G is "$k'^{\times}$ viewed as a k-group".) This canonically contains GL_1 as a k-subgroup.

The smooth connected quotient G/GL_1 of dimension $p^n - 1 > 0$ is killed by the p^n-power map since $k'^{p^n} \subseteq k$, so it is unipotent and hence G is not reductive. (More explicitly, $G_{\overline{k}}$ is the algebraic group of units of $k' \otimes_k \overline{k}$, in which the subgroup of 1-units is a codimension-1 unipotent radical.) However, $G(k_s) = (k' \otimes_k k_s)^\times$ has no nontrivial p-torsion since $k' \otimes_k k_s$ is a field of characteristic p, so the smooth connected commutative unipotent k-subgroup $\mathscr{R}_{u,k}(G)$ must be trivial. That is, G is pseudo-reductive over k. The k-group G naturally occurs (up to conjugacy) inside of GL_{p^n}. A non-commutative generalization of this example is given in Example 1.1.11.

Remark 1.1.4 Let G be a smooth connected affine group over a field k. It is elementary to prove that $G/\mathscr{R}_{u,k}(G)$ is pseudo-reductive over k. In particular, G is canonically an extension of the pseudo-reductive k-group $G/\mathscr{R}_{u,k}(G)$ by the smooth connected unipotent k-group $\mathscr{R}_{u,k}(G)$. To use this extension structure to reduce problems for G to problems for pseudo-reductive k-groups, we need to know something about the structure of smooth connected unipotent k-groups.

A general smooth connected unipotent k-group U is hard to describe (especially when $\mathrm{char}(k) > 0$), but any such U admits a characteristic central composition series whose successive quotients are k-forms V_i of vector groups $\mathbf{G}_\mathrm{a}^{n_i}$ (by [SGA3, XVII, 4.1.1(iii)], or by Corollary B.2.7, Corollary B.3.3, and Theorem B.3.4 when $\mathrm{char}(k) > 0$). We have $V_i \simeq \mathbf{G}_\mathrm{a}^{n_i}$ when k is perfect (by [SGA3, XVII, 4.1.5], or Corollary B.2.7 when $\mathrm{char}(k) > 0$) but V_i is mysterious in general if k is imperfect and $U \neq 1$. Nonetheless, such V_i are commutative and p-torsion when $\mathrm{char}(k) = p > 0$, so there are concrete hypersurface models for V_i over infinite k (Proposition B.1.13). This often makes $\mathscr{R}_{u,k}(G)$ tractable enough so that problems for general G can be reduced to the pseudo-reductive case.

There are interesting analogies between pseudo-reductive k-groups and connected reductive k-groups. For example, we will prove by elementary methods that the Cartan k-subgroups (i.e., centralizers of maximal k-tori) in pseudo-reductive k-groups are always commutative and pseudo-reductive; see Proposition 1.2.4. We do not know an elementary proof of the related fact that, when $\mathrm{char}(k) \neq 2$, a pseudo-reductive k-group is reductive if and only if its Cartan k-subgroups are tori. Our proof of this result (Theorem 11.1.1) rests on our main structure theorem for pseudo-reductive groups. It seems unlikely that an alternative proof can be found (bypassing the structure theorem), since the avoidance of characteristic 2 is essential. Indeed, we will show in Example 11.1.2 that over every imperfect field k with characteristic 2 there are non-reductive pseudo-reductive k-groups G whose Cartan k-subgroups are tori.

In view of the commutativity and pseudo-reductivity of Cartan subgroups in pseudo-reductive groups, a basic reason that the structure of pseudo-reductive groups is more difficult to understand than that of connected reductive groups is that we do not understand the structure of the commutative objects very well. For example, whereas k-tori are unirational over k for any field k, in Example 11.3.1 we will exhibit commutative pseudo-reductive k-groups that are not unirational over k, where k is any imperfect field.

Example 1.1.5 The property of pseudo-reductivity over arbitrary imperfect fields exhibits some behavior that is not at all like the more familiar connected reductive case. We mention two such examples now, the second of which is more interesting than the first.

(i) Pseudo-reductivity can be destroyed by an inseparable ground field extension, such as scalar extension to the perfect closure. This is not a surprise, since by definition it happens for every non-reductive pseudo-reductive group (of which we shall give many examples).

In contrast, by Proposition 1.1.9(1) below, pseudo-reductivity is insensitive to separable extension of the ground field. This will often be used when passing to a separably closed ground field in a proof. An arithmetically interesting example of a non-algebraic separable extension is k_v/k for a global function field k and a place v of k. Thus, a group scheme G over a global function field k is pseudo-reductive over k if and only if G_{k_v} is pseudo-reductive over k_v.

(ii) The second, and more surprising, deviation of pseudo-reductivity from reductivity is that it is generally not inherited by quotients. For example, consider a non-reductive pseudo-reductive group G over a field k with $\mathrm{char}(k) = p > 0$. By Galois descent, the subgroup $\mathscr{R}_u(G_{\overline{k}})$ in $G_{\overline{k}}$ is defined over the perfect closure k_{p} of k. Since k_{p} is perfect, the k_{p}-descent of $\mathscr{R}_u(G_{\overline{k}})$ has a composition series over k_{p} whose successive quotients are k_{p}-isomorphic to $\mathbf{G}_{\mathrm{a}}$ (Proposition A.1.4). But k_{p} is the direct limit of subfields $k^{p^{-n}}$, so if n is sufficiently large then $\mathscr{R}_u(G_{\overline{k}})$ descends to a nontrivial unipotent subgroup U of $G_{k^{p^{-n}}}$ such that U is $k^{p^{-n}}$-split. The p^n-power map identifies $k^{p^{-n}}$ with k carrying the inclusion $k \hookrightarrow k^{p^{-n}}$ inside of $\overline{k}$ over to the p^n-power map of k, so $G_{k^{p^{-n}}}$ is thereby identified with the target of the n-fold Frobenius isogeny $F_{G/k,n} : G \to G^{(p^n)}$ as in Definition A.3.3.

Hence, *every* non-reductive pseudo-reductive k-group G is a (purely inseparable) isogenous cover of a smooth connected affine k-group G' such that the smooth connected unipotent k-group $\mathscr{R}_{u,k}(G')$ is nontrivial and is a successive extension of copies of $\mathbf{G}_{\mathrm{a}}$ over k. Such examples are of limited interest since the structure of $\ker(G \twoheadrightarrow G')$ is not easily understood. But there are more interesting examples of pseudo-reductive k-groups G admitting quotients G/H that are not pseudo-reductive. This was illustrated in Example 1.1.3 in the commutative case with H a torus, and we will see examples using (infinitesimal) finite central multiplicative H (Example 1.3.2 for commutative G and Example 1.3.5 for perfect G) as well as very surprising examples using perfect smooth connected H and perfect G (Example 1.6.4). These examples exist over any imperfect field.

An important ingredient in our work is the minimal intermediate field in $\overline{k}/k$ over which the geometric unipotent radical $\mathscr{R}_u(G_{\overline{k}})$ inside of $G_{\overline{k}}$ is defined. This is a special case of a useful general notion:

Definition 1.1.6 Let X be a scheme over a field k, K/k an extension field, and $Z \subseteq X_K$ a closed subscheme. For an intermediate field k'/k, Z is *defined over* k' if Z descends (necessarily uniquely) to a closed subscheme of $X_{k'}$. The (minimal) *field of definition* (over k) for $Z \subseteq X_K$ is the unique such k'/k that is contained in all others. (For a thorough discussion of the general existence of such a field, we refer the reader to [EGA, IV$_2$, §4.8], especially [EGA, IV$_2$, 4.8.11].)

Sometimes for emphasis we append the word "minimal" when speaking of fields of definition, but in fact minimality is always implicit.

Remark 1.1.7 The mechanism underlying the existence of (minimal) fields of definition of closed subschemes is a fact from linear algebra, as follows. Consider an extension of fields K/k, a k-vector space V, and a K-subspace $W \subseteq V_K$. Among all subfields $F \subseteq K$ over k such that W arises by scalar extension from an F-subspace of V_F, we claim that there is one such F that is contained in all others.

To see that such a minimal field exists, choose a k-basis $\{e_i\}_{i\in I}$ of V and a subset $B = \{e_i\}_{i\in J}$ projecting to a K-basis of $(K \otimes_k V)/W$. Then F is generated over k by the coefficients of the vectors $\{e_i \bmod W\}_{i\notin J}$ relative to the K-basis B. We call F the *field of definition* over k for the K-subspace W.

The following lemma records the simple behavior of fields of definition under a ground field extension.

Lemma 1.1.8 *Let k be a field, $L'/k'/k$ a tower of extensions, and L an intermediate field in L'/k. Let X be a k-scheme and Y a closed subscheme of $X_{k'}$, and $F \subseteq k'$ the minimal field of definition over k for Y. The minimal field of definition over L for the closed subscheme $Y_{L'} := Y \otimes_{k'} L'$ in $X_{k'} \otimes_{k'} L' = X_L \otimes_L L'$ is the compositum FL inside of L'.*

In particular, if $L' = Lk'$ and $Y \subseteq X_{k'}$ has minimal field of definition over k equal to k' then the closed subscheme $Y_{L'} \subseteq X_{L'}$ has minimal field of definition over L equal to L'.

The most useful instance of this lemma for our purposes will be the case when k'/k is purely inseparable and L/k is separable (with $F = k'$) but no ambient L' is provided. In such cases $L \otimes_k k'$ is a field (as we see by passage to the limit from the case when k'/k is finite) and so we can take $L' := L \otimes_k k'$.

Proof. We first reduce to the affine case, using a characterization of the minimal field of definition F/k that is better suited to Zariski-localization: if $\{U_i\}$ is an open cover of X (e.g., affine opens) then a subfield K of k' over k contains F if and only if the closed subscheme $Y \cap (U_i)_{k'} \subseteq (U_i)_{k'}$ is defined over K for all i. (This follows from the uniqueness of descent of a closed subscheme to a minimal field of definition.) In other words, F is the compositum of the fields of definition over k for each $Y \cap (U_i)_{k'} \subseteq (U_i)_{k'}$. Since the field of definition for $Y_{L'} \subseteq X_{L'}$ over L is the corresponding compositum for the closed subschemes $Y_{L'} \cap (U_i)_{L'} \subseteq (U_i)_{L'}$, if we can handle each U_i separately then passage to composite fields gives the global result. Thus, we now may and do assume X is affine, say $X = \operatorname{Spec} A$ for a k-algebra A.

The ideal of Y in $A_{k'}$ has the form $k' \otimes_F J$ for an ideal J in A_F, and F is the smallest intermediate extension in k'/k to which $k' \otimes_F J$ descends as an ideal. But descent as an ideal is equivalent to descent as a vector subspace (as the property of a subspace of an algebra being an ideal can be checked after an extension of the ground field), so F/k is the minimal field of definition over k for the k'-subspace $V = k' \otimes_F J$ inside of $A_{k'}$.

The proof of existence of minimal fields of definition for subspaces of a vector space (as in Remark 1.1.7) implies that F is also the minimal field of definition over k for the L'-subspace $L' \otimes_{k'} V \subseteq A_{L'}$. Let K be the minimal field of definition over L for this L'-subspace. Our aim is to prove that $K = FL$. Since the L'-subspace $L' \otimes_{k'} V$ is defined over FL (as it even descends to the F-subspace J in A_F), we have $L \subseteq K \subseteq FL$. The problem is therefore to prove that $F \subseteq K$. But $L' \otimes_{k'} V$ does descend to a K-subspace of A_K by definition of K/L, so by the minimality property for F relative to L'/k we get $F \subseteq K$. □

Proposition 1.1.9 *Let K/k be a separable extension of fields, and G a smooth connected affine k-group.*

(1) *Inside of G_K we have $\mathscr{R}_{u,k}(G)_K = \mathscr{R}_{u,K}(G_K)$. In particular, G is pseudo-reductive over k if and only if G_K is pseudo-reductive over K.*
(2) *Choose a k-embedding $\overline{k} \hookrightarrow \overline{K}$ of algebraic closures. The field of definition E_k over k for $\mathscr{R}_u(G_{\overline{k}}) \subseteq G_{\overline{k}}$ is a finite purely inseparable extension and $K \otimes_k E_k = E_K$ inside of $\overline{K}$.*

The analogue of Proposition 1.1.9(1) for the maximal k-split smooth connected unipotent normal k-subgroup $\mathscr{R}_{us,k}(G)$ lies deeper, and is given in Corollary B.3.5.

Proof. First we prove (1). By Galois descent, if k'/k is a Galois extension then $\mathscr{R}_{u,k'}(G_{k'})$ descends to a smooth connected unipotent normal k-subgroup of G. Such a descent is contained in $\mathscr{R}_{u,k}(G)$, so this settles the case where K/k is Galois. Applying this to separable closures k_s/k and K_s/K (with K_s chosen to contain k_s over k), we may assume that k is separably closed. Since $\mathscr{R}_{u,k}(G)_K \subseteq \mathscr{R}_{u,K}(G_K)$ in general, to prove the equality as in (1) it suffices to prove an inequality of dimensions in the opposite direction when $k = k_s$.

More generally, if U is a smooth connected unipotent normal K-subgroup of G_K, say with $d = \dim U \geqslant 0$, we shall apply a specialization argument to U to construct a smooth connected unipotent normal k-subgroup of G with dimension d when $k = k_s$. By expressing K as a direct limit of its finitely generated subfields over k, there is such a subfield F for which U descends to an F-subgroup of G_F that is necessarily smooth, connected, unipotent, and normal in G_F. Thus, upon renaming F as K we may assume that K/k is finitely generated. Separability of K/k then allows us to write $K = \mathrm{Frac}(A)$ for a k-smooth domain A.

The normality of U in G can be expressed as the fact that the map $G_K \times U \to G_K$ defined by $(g, u) \mapsto gug^{-1}$ factors through U, and the unipotence can be expressed as the fact that for some finite extension K'/K the K'-group $U_{K'}$ admits a composition series whose successive quotients are $\mathbf{G}_{\mathrm{a}}$. Since K is the direct limit of its k-smooth subalgebras $A[1/a]$ for $a \in A - \{0\}$, by replacing A with a suitable such $A[1/a]$ we may arrange that $U = \mathscr{U}_K$ for a closed subscheme $\mathscr{U} \subseteq G_A$.

Let A' be an A-finite domain such that $A'_K = K'$. By "spreading out" of properties of $U = \mathscr{U}_K$ and $\mathrm{Spec}\, K' = (\mathrm{Spec}\, A')_K$ from the fiber over the generic point $\mathrm{Spec}\, K$ of $\mathrm{Spec}\, A$, upon replacing A with a further localization $A[1/a]$ we can arrange that $\mathscr{U}$ is an A-smooth normal A-subgroup of G_A with (geometrically) connected fibers of dimension d, and that there is a finite faithfully flat extension $A \to A'$ (with generic fiber K'/K) such that $\mathscr{U}_{A'}$ admits a composition series by A'-smooth normal closed A'-subgroups with successive quotients isomorphic to $\mathbf{G}_{\mathrm{a}}$ as an A'-group. Hence, all fibers of $\mathscr{U}$ over $\mathrm{Spec}\, A$ are unipotent. Since A is k-smooth, if $k = k_s$ then there exist k-points of $\mathrm{Spec}\, A$. The fiber of $\mathscr{U}$ over such a point is a smooth connected unipotent normal k-subgroup of G with dimension d. This proves (1).

For (2), we apply Lemma 1.1.8 to the tower $\overline{K}/\overline{k}/k$, the intermediate field K, and the k-scheme $X = G$ and $\overline{k}$-scheme $Y = \mathscr{R}_u(G_{\overline{k}})$ (for which $Y_{\overline{K}} = \mathscr{R}_u(G_{\overline{K}})$). The assertion of the Lemma in this case is that the field of definition E_K over K for $\mathscr{R}_u(G_{\overline{K}})$ is the composite field $E_k K$ inside of $\overline{K}$. Provided that E_k/k is purely inseparable, the ring $E_k \otimes_k K$ is a field (since K/k is separable) and hence the surjective map $E_k \otimes_k K \to E_k K$ is an isomorphism. It therefore remains to prove that E_k/k is a purely inseparable extension of finite degree. By Galois descent, $\mathscr{R}_u(G_{\overline{k}})$ is defined over the perfect closure $k_{\rm p}$ of k, which is to say that it descends to a $k_{\rm p}$-subgroup $U \subseteq G_{k_{\rm p}}$. Since the ideal of U in the coordinate ring of $G_{k_{\rm p}}$ is finitely generated, and every finite subset of $k_{\rm p}$ is contained in a subextension of finite degree over k, we can descend U to a k'-subgroup of $G_{k'}$ for some finite purely inseparable extension k'/k. Necessarily $E_k \subseteq k'$, so we are done. $\square$

To make interesting non-commutative pseudo-reductive groups, we need to study the Weil restriction of scalars functor $\mathrm{R}_{k'/k}$ applied to connected reductive groups over finite extension fields k'/k. For later purposes, it is convenient to work more generally with nonzero finite reduced k-algebras k', which is to say $k' = \prod k'_i$ for a non-empty finite collection $\{k'_i\}$ of fields of *finite* degree over k. This generality is better-suited to Galois descent (which we use very often): the functor $(\cdot) \otimes_k k_s$ carries nonzero finite reduced k-algebras to nonzero finite reduced k_s-algebras, but generally does not carry fields to fields. For such general $k' = \prod k'_i$, a quasi-projective k'-scheme X' is precisely $\coprod X'_i$ for quasi-projective k'_i-schemes X'_i, and $\mathrm{R}_{k'/k}(X') = \prod \mathrm{R}_{k'_i/k}(X'_i)$.

Proposition 1.1.10 *Let k be a field, k' a nonzero finite reduced k-algebra, and G' a k'-group whose fibers over* Spec k' *are connected reductive* (*or more generally, pseudo-reductive*). *The smooth connected affine k-group $G = \mathrm{R}_{k'/k}(G')$ is pseudo-reductive.*

Proof. By Proposition A.5.2(4) and Proposition A.5.9, the affine k-group G is smooth and connected. Let $\iota : U \hookrightarrow G$ be a smooth connected unipotent normal k-subgroup of G. We must show that ι is the trivial map (i.e., $U = 1$). It is equivalent to prove triviality of the k'-map $\iota' : U_{k'} \to G'$ that corresponds to ι via the universal property of Weil restriction. The image $H' \subseteq G'$ of ι' is a smooth unipotent k'-subgroup of G' with connected fibers over Spec k', and we claim that it is normal. Once this is proved, then reductivity (or even just pseudo-reductivity) of the fibers of G' over Spec k' implies that ι' is trivial, so G is indeed pseudo-reductive over k.

To verify the normality of H' in G', we first observe that (by construction) ι' is the restriction to $U_{k'}$ of the canonical map $q : G_{k'} \to G'$ that corresponds to the identity map of $G = \mathrm{R}_{k'/k}(G')$ under the universal property of Weil restriction. Smoothness of G' implies that q is surjective (Proposition A.5.11(1)), so normality of $U_{k'}$ in $G_{k'}$ implies that the k'-group $H' = q(U_{k'})$ is normal in G'. $\square$

The k-group $G = \mathrm{R}_{k'/k}(G')$ for a k'-group G' with connected reductive fibers is especially interesting when G' has a nontrivial fiber over some factor field k'_i

of k' that is not separable over k, because then the pseudo-reductive k-group G is *not* reductive. This will be proved in Example 1.6.1. Here is a concrete instance of this fact.

Example 1.1.11 Consider a finite extension of fields k'/k and a central simple k'-algebra A' with $\dim_{k'} A' = n^2$. The k'-group G' of units of A' is a connected reductive k'-group (a k'-form of GL_n). The k-group $G := \mathrm{R}_{k'/k}(G')$ of units of the finite-dimensional simple (but non-central when $k' \neq k$) k-algebra underlying A' is pseudo-reductive, by Proposition 1.1.10. When k'/k is not separable, the unit group G is not reductive over k; this is a special case of Example 1.6.1.

Example 1.1.12 Let k'/k be a finite extension of fields and G' a connected reductive k'-group. Let k_1/k be the maximal separable subextension in k'/k (so k'/k_1 is purely inseparable), and k_s/k a separable closure. Since $\mathrm{R}_{k'/k} = \mathrm{R}_{k_1/k} \circ \mathrm{R}_{k'/k_1}$, the decomposition of $k_1 \otimes_k k_s$ into a finite product of copies of k_s gives rise to the decomposition

$$\mathrm{R}_{k'/k}(G')_{k_s} \simeq \prod_{\sigma} (\mathrm{R}_{k'/k_1}(G') \otimes_{k_1,\sigma} k_s), \tag{1.1.1}$$

with σ ranging through the k-embeddings of k_1 into k_s. This often reduces general problems for $\mathrm{R}_{k'/k}(G')$ to the special case of purely inseparable extension fields.

1.2 Elementary properties of pseudo-reductive groups

We now prove some general properties of pseudo-reductive groups that follow from the definition of pseudo-reductivity (and from general facts in the theory of reductive groups). To begin, we record a lemma that is extremely useful in many arguments with pseudo-reductivity.

Lemma 1.2.1 *Let G be a pseudo-reductive k-group, and X a closed subscheme of G such that $X_{\overline{k}} \subseteq \mathscr{R}_u(G_{\overline{k}})$ and $X_{\overline{k}}$ is reduced and irreducible. Then X is a k-rational torsion point in the center of G, trivial if* $\mathrm{char}(k) = 0$ *and with p-power order if* $\mathrm{char}(k) = p > 0$. *In particular, if X contains the identity then it is the identity point.*

Note that the center of G contains only finitely many k-rational p-power torsion points when $\mathrm{char}(k) = p > 0$. Indeed, the Zariski closure of the group of such points is a smooth unipotent central k-subgroup of G, so its identity component is normal in G and thus is trivial.