1 Numbers and indices

Most of this chapter should be familiar, but it is important that you really understand all of the material, which is largely a series of definitions.

1.1 Numbers

Real numbers are numbers that can be fitted into a place on the number scale (Fig. 1.1). The other kinds of numbers are **complex** (or **imaginary**) **numbers**, which cannot be fitted onto this scale, but lie above or below the line. They are of the general form $a + ib$, where a and b are real numbers but i is the square root of –1.

Real numbers can be divided into:

Integers: these are whole numbers, positive or negative, such as 7, 341, –56.

Rational numbers: these can be expressed *precisely* as the ratio of two integers. All integers are rational (they can be written as n / 1) and many non-integers are also rational, such as 3 / 4, 2.5 (= 5 / 2), –7.36 (= –736 / 100).

Irrational numbers: these *cannot be precisely* expressed as the ratio of two integers; examples are π (which is not exactly 22 / 7 nor any other ratio of integers) and the square roots of all prime numbers (except 1). Note that a number that has to be written as a recurring decimal is not irrational: 0.333 333 … is exactly 1 / 3; and 0.142 857 142 857 142 857 … is 1 / 7. Also, all approximations are rational: if we give π the approximate value of 3.142 this is 3142 / 1000, a rational number.

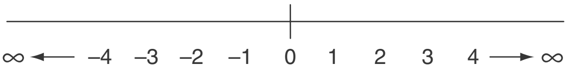

Fig. 1.1 The scale of real numbers.

1.2 Indices

A number written as n^a is defined as **the number *n* raised to the power *a***. If a is a positive integer (the simplest case) then n^a means that n is multiplied a times by itself (n). Thus, 2^5 means $2 \times 2 \times 2 \times 2 \times 2 = 32$. In the expression n^a the number n is called the **base**, and a is called the **index** (or power, or exponent). Neither n nor a need to be integers.

Expressions that include more than one base cannot always be simplified: nothing for instance can be done with an expression such as $n^a \times z^b$. However, if the power is the same, we may be able to rewrite, as for example, $n^a \times z^a = (nz)^a$. Simplifications are possible in some very important cases where only one base is present, as follows.

Multiplication

$$n^a \times n^b = n^{(a+b)}$$

$$\text{e.g. } 5^2 \times 5^3 = (5 \times 5) \times (5 \times 5 \times 5) = 5^5 = 25 \times 125 = 3125$$

Note carefully that this only works when the base is constant. Expressions such as $n^a \times z^b$ cannot be treated in this way.

Division

$$n^a / n^b = n^{(a-b)}$$

$$\text{e.g. } 2^4 / 2^3 = (2 \times 2 \times 2 \times 2)/(2 \times 2 \times 2) = 2^1 = 2$$

Again the base must be constant for this to work.

These two relationships are the foundation for the use of logarithms (which are themselves indices of a chosen base number) as aids to multiplication and division.

Powers of indices

$$(n^a)^b = n^{a \times b}$$

$$\text{e.g. } (3^2)^3 = 3^2 \times 3^2 \times 3^2 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6 = 729$$

Note carefully that expressions such as $n^a + n^b$ or $n^a - n^b$ cannot be simplified (unless the actual numerical values of n, a and b are known).

$$\begin{aligned} \text{e.g. } 3^3 + 3^1 &= (3 \times 3 \times 3) + 3 = 27 + 3 \\ &= 30 \quad \text{and not } 3^4 \text{ (which equals 81)}, \\ 3^3 - 3^2 &= (3 \times 3 \times 3) - (3 \times 3) \\ &= 27 - 9 = 18 \quad \text{and not } 3^1 \text{ (which equals 3)} \end{aligned}$$

Indices need not be only positive integers. They may also have zero value (e.g. n^0), or be negative (e.g. n^{-3}) or fractional (e.g. $n^{3/2}$). It is important to understand what these different usages mean.

Any base raised to the power 0 has a value of 1:

$$n^0 = 1$$

$$(n^a \div n^a = 1 = n^{(a-a)} = n^0)$$

Any base raised to the power 1 has a value equal to the base itself:

$$n^1 = n$$

A base raised to the power 0.5 has a value equal to the square root of the base:

$$n^{0.5} \times n^{0.5} = n^1$$

A base raised to a negative power represents the reciprocal of the base raised to that same (but positive) power:

$$n^{-a} = 1/n^a$$

The value of $n^{a/b}$ is the bth root of n^a:

$$n^{a/b} = \sqrt[b]{n^a}$$

So, for example, $2^{5/4} = \sqrt[4]{2^5} = 2^{1.25}$ and $3^{-3/2} = 1/\sqrt[2]{3^3} = 1/3^{1.5}$. These kinds of expression are most easily solved by using logarithms (or a pocket calculator), as will be discussed later (see Chapter 5).

Standard form

In order to write big (or small) numbers in a compact way we express them as powers of 10, for example:

$$234\,700\,000 = 2.347 \times 10^8;\ \ 0.000\,000\,625 = 6.25 \times 10^{-7}$$

While these numbers could just as well be written as 23.47×10^7 and 0.625×10^{-6}, the standard form is to show a single integer (other than 0) to the left of the decimal point.

Two things to be careful about:

(1) Going from standard form to a written-out number can be treacherous and so take **great** care. 2×10^{-3} does not equal 0.02: rather, $2 \times 10^{-3} = 0.002$. This seems obvious yet this kind of error is common.

(2) You can add or subtract numbers in standard form **only** when all the numbers are rewritten each at the same power of 10. At the end you can convert back to standard form if necessary.

$$\begin{aligned}(3 \times 10^3) + (8 + 10^2) - (5 \times 10^1) &= (300 \times 10^1) + (80 \times 10^1) \\ &\quad -(5 \times 10^1) \\ &= 375 \times 10^1 \\ &= 3.75 \times 10^3\end{aligned}$$

Getting this right looks easy but is really quite troublesome. An example on the Internet that shows how to do this kind of calculation is worked out to the wrong answer! The safest thing is to write out all the numbers fully (i.e. as multiplied by 10^0):

$$3000 + 800 - 50 = 3750 = 3.75 \times 10^3$$

Multiplying (or dividing) numbers in standard form is relatively easy:

$$\begin{aligned}(3 \times 10^3) \times (8 \times 10^2) \times (5 \times 10^1) &= 3 \times 8 \times 5 \times 10^{(3+2+1)} \\ &= 120 \times 10^6 \\ &= 1.20 \times 10^8 \\ (8 \times 10^2) \div (5 \times 10^1) &= (8 \div 5) \times 10^{(2-1)} = 1.6 \times 10^1\end{aligned}$$

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2 A sense of proportion

If the Eiffel tower were now representing the world's age, the skin of paint on the pinnacle-knob at its summit would represent man's share of that age, and anybody would perceive that the skin was what the tower was built for. I reckon they would, I dunno.

Mark Twain

The object of this chapter is to encourage you to think whether or not your answer to a problem looks reasonable or ridiculous. In general, a reasonable answer is likely to be a right answer. An answer that looks ridiculous might also be right, but you should then be alert to check your calculation very carefully. Of course, there will be times when you do not know what to make of an answer – is it reasonable or is it not? The better your background of knowledge and experience, the less often will this uncertainty happen.

2.1 A ridiculous answer that is wrong

Here is the problem: calculate what dry weight of bacteria will be present in 10 litres of medium in a fermenter after 10 h when at time zero there are 10 organisms ml^{-1} and there is a lag of 1 h before exponential growth (doubling time 20 min) begins. One organism has a dry weight of 1×10^{-12} g.

This is the answer from a candidate in an examination (examiner's comments in [ ]*):*

There are 9 hours of exponential growth

In 1 hour there are 3 doublings ($t_d = 20$ min)

Therefore there are 27 doublings in total

So that 10×2^{27} organisms will be present per ml after 10 hours [*perfect so far*]

$= 2 \times 10^{27}$ organisms per ml [*spectacularly wrong; needs to read about indices*]

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$\equiv 2 \times 10^{27} \times 1 \times 10^{-12} \times 10^{4}$ g dry weight in 10 litres [*right if line above were right*]

$= 2 \times 10^{19}$ g

Now the candidate starts to worry. [*The sadistic examiner begins to be amused*]

$= 2 \times 10^{16}$ kg

[*The examiner is laughing*]

$= 2 \times 10^{13}$ metric tonnes

This would not fit in the fermenter, *writes the candidate as the last line of answer.*

[*The examiner is rolling on the floor; marking scripts has its compensations*]

This answer is plainly ridiculous, but it is not clear whether the candidate has realised there is a mistake in the calculation, or (more probably) whether the examiner is being implicitly criticised for setting a problem that has a stupid answer. As this latter circumstance never happens (well, hardly ever), then there must be a mistake, as is pointed out above:

10×2^{27} does not equal 2×10^{27}. Rather, $10 \times 2^{27} = 10 \times 1.342 \times 10^{8}$

The correct dry weight after 10 hours is $1.342 \times 10^{9} \times 1 \times 10^{-12}$ g ml^{-1} = 1.342×10^{-3} g ml^{-1} or 1.342 mg ml^{-1} and so 13.42 g in 10 L

This answer does not appear impossible, and looks plausible if one has some knowledge of the levels of growth that bacterial cultures typically reach (1 to 10 mg dry weight ml^{-1}).

Simple mistakes in calculation are the commonest reason for getting wrong answers. Always think about the likely size of a result, and be sure to get ratios the right way round. For example, if you are finding how much of an anhydrous compound to use in a solution, when the recipe calls for a hydrated salt, then the required amount will be smaller than the recipe says. Remembering that where x is a positive real number:

multiplying x by a positive number less than 1 will lead to a number smaller than x
multiplying x by a number bigger than 1 will lead to a number bigger than x
dividing x by a positive number less than 1 will lead to a number bigger than x
dividing x by a number bigger than 1 will lead to a number smaller than x

should help you to express proportions the correct way round.

2.2 'Back of envelope' calculations

One of the most useful things you may learn from this book is how to get an approximate idea of the answer to a calculation by doing quite drastic rounding up and rounding down of the numbers in an expression. For example:

2875×7681 can be rounded to 3000×7500 which is 22 500 000
(If one number is rounded up, try to round down another.)
The precise answer is 22 082 875, and so the approximation is only off by 1.9%.

Here is another: $(3478 \times 29\,641) / (391 \times 475)$ can be written as:

$$\frac{3478 \times 29\,641}{391 \times 475} \longrightarrow \frac{3500 \times 30\,000}{400 \times 500}$$

$$\frac{\overset{175}{\cancel{\overset{700}{\cancel{3500}}}} \times \cancel{30\,000}}{\underset{1}{\cancel{400}} \times \underset{1}{\cancel{500}}} \longrightarrow 525$$

Rounding the numbers allows drastic cancelling, to give an answer only 5.4% away from the precise result, which is 555.1.

Even if you can do this more quickly with a calculator, you can also easily make mistakes in pressing wrong keys, and for many people disbelieving what the calculator says is difficult. The last example (below) is none too simple with a calculator, and back-of-envelope work is highly desirable to get an idea of what to expect as the answer.

$$\frac{(7836-484) \times 9741}{(2743 \times 37)+960} \longrightarrow \frac{(8000-500) \times 10\,000}{(3000 \times 40)+1000}$$

$$\longrightarrow \frac{7500 \times 10\,000}{120\,000 + 1000} \longrightarrow \frac{7500 \times 10\,000}{120\,000}$$

$$\longrightarrow \frac{\overset{2500}{\cancel{7500}} \times 1\cancel{0\,000}}{\underset{4}{\cancel{120\,000}}} \longrightarrow 625$$

The precise answer is 699.0, which means that this time the rough answer is not so close; the error is 10.6%, but even so this still gives a good idea of what to expect as the correct result after accurate calculation.

You probably think that all these examples were carefully devised, with the roundings planned ahead to get a good answer. They were in fact done with no forethought of that kind, and are genuine, honestly. Doing back-of-envelope calculations without an envelope (i.e. in your head) is a talent that can be shown off to the uninitiated (impress your friends!), but be very careful not to lose track of powers of 10. The envelope is safer.

3 Graphs

'and what is the use of a book,' thought Alice, 'without pictures or conversation?'

Lewis Carroll

Why draw a graph? There are many reasons, but the fundamental one is that the human brain understands a picture much more easily than it does a table of numbers.

Many data-handling questions require a graph to be drawn as part of the solution. It is unlikely that under examination conditions a work of art will be produced, nor would one be expected. However, some marks are given for a graph that is correct (the points are plotted in the right places!) and which obeys the conventional rules.

As well as making the drawing, you will probably have to use the graph to read off some values, such as a gradient or an intercept or to measure test samples from an assay. Doing these interpretations will be considered after discussing how to produce a graph.

3.1 Drawing graphs

The graph shown in Fig. 3.1 illustrates a number of features.

There are several things to note. The horizontal scale (x axis, or abscissa) is given to the variable that is the more directly under the control of the investigator, and the variable that is measured for various values of x is plotted on the vertical scale (y axis, or ordinate). In Fig. 3.1, the times at which readings of the extinction are made are chosen by the experimenter, and so go on the x axis, while the extinctions themselves are less under control and follow from the selected times, and therefore go on the y axis.

Do not make the graph too small; aim to use as much of the area of the sheet of graph paper as possible. The scales of the two axes must therefore be chosen with care. Neither scale should extend far beyond the plotted points,

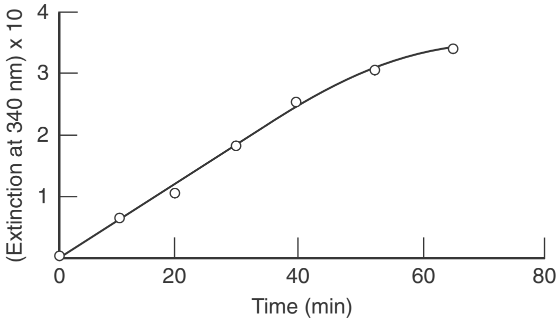

Fig. 3.1 Reduction of NAD^+ by lactate dehydrogenase. At time zero, enzyme, substrate and cofactor were mixed and extinctions at 340 nm (due to formation of NADH) were measured at 30 °C at intervals. Readings were continued for 66 min from the time of mixing, and the measured extinctions fell in the range 0 to 0.32.

and the points themselves must always lie within the scales. Straight-line graphs are best plotted with scales devised in such a way that the line makes an angle of about 45° with the *x* axis, so that *x* or *y* values can be plotted or read from the graph with similar precision. (When several lines are plotted on one graph it is unlikely that this rule can be obeyed for all the lines, but at least one of the lines ought to be plotted to best advantage.)

The origin of a graph does not necessarily have to be shown. Frequently a better graph can be made by using scales of limited ranges. Compare Fig. 3.2a and b.

A problem sometimes occurs in the laboratory with real experimental results: '*Must* a straight line be drawn to go through the origin even though doing this gives a line of poorer fit with the data?' Unfortunately, the answer is sometimes 'Yes' and sometimes 'No' depending on all kinds of things. Fortunately the made-up results given in a data-handling question will not be equivocal (unless you are specially warned!) and will not lead to points on graphs that leave much doubt about where the curve ought to go, that is, provided the points are plotted in the right places. Real-life problems may not be so amenable!

Frequently points are not plotted correctly. People often miscount squares on graph paper and hence make scales with irregular spacing of the scale divisions. Another common error is to choose a logarithmic scale when a linear one ought to have been used. If you have numbers spread between 10 and 10 000 to plot (e.g. organisms per ml), and you label the