An Introduction to Random Matrices

The theory of random matrices plays an important role in many areas of pure mathematics and employs a variety of sophisticated mathematical tools (analytical, probabilistic and combinatorial). This diverse array of tools, while attesting to the vitality of the field, presents several formidable obstacles to the newcomer, and even the expert probabilist.

This rigorous introduction to the basic theory is sufficiently self-contained to be accessible to graduate students in mathematics or related sciences, who have mastered probability theory at the graduate level, but have not necessarily been exposed to advanced notions of functional analysis, algebra or geometry. Useful background material is collected in the appendices and exercises are also included throughout to test the reader’s understanding. Enumerative techniques, stochastic analysis, large deviations, concentration inequalities, disintegration and Lie algebras all are introduced in the text, which will enable readers to approach the research literature with confidence.

Greg W. Anderson is Professor of Mathematics at the University of Minnesota.

Alice Guionnet is CNRS Research Director at the Ecole Normale Supérieure in Lyon (ENS-Lyon).

Ofer Zeitouni is Professor of Mathematics at both the University of Minnesota and the Weizmann Institute of Science in Rehovot, Israel.
CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: www.cambridge.org/series/sSeries.asp?code=CSAM

Already published

65 A. J. Berrick & M. E. Keating An introduction to rings and modules with K-theory in view
66 S. Morosawa et al. Holomorphic dynamics
67 A. J. Berrick & M. E. Keating Categories and modules with K-theory in view
68 K. Sato Lévy processes and infinitely divisible distributions
69 H. Hida Modular forms and Galois cohomology
70 R. Iorio & V. Iorio Fourier analysis and partial differential equations
71 R. Blei Analysis in integer and fractional dimensions
72 F. Borceux & G. Janelidze Galois theories
73 B. Bollobás Random graphs (2nd Edition)
74 R. M. Dudley Real analysis and probability (2nd Edition)
75 T. S. Shil-Complex polynomial
76 C. Voisin Hodge theory and complex algebraic geometry, I
77 C. Voisin Hodge theory and complex algebraic geometry, II
78 V. Paulsen Completely bounded maps and operator algebras
79 F. Gesztesy & H. Holden Soliton equations and their algebro-geometric solutions, I
80 S. Mukai An introduction to invariants and moduli
81 G. Tourlakis Lectures in logic and set theory, I
82 G. Tourlakis Lectures in logic and set theory, II
83 R. A. Bailey Association schemes
84 J. Carlson, S. Müller-Stach & C. Peters Period mappings and period domains
85 J. J. Duistermaat & J. A. C. Kolk Multidimensional real analysis, I
86 J. J. Duistermaat & J. A. C. Kolk Multidimensional real analysis, II
87 M. C. Golumbic & A. N. Trenk Tolerance graphs
88 L. H. Harper Global methods for combinatorial isoperimetric problems
89 J. M. J. Dumas & J. Meun Introduction to foliations and Lie groupoids
90 J. Kolár, K. E. Smith & A. Corti Rational and nearly rational varieties
91 D. Applebaum Lévy processes and stochastic calculus (1st Edition)
92 B. Conrad Modular forms and the Ramanujan conjecture
93 M. Schechter An introduction to nonlinear analysis
94 R. Carter Lie algebras of finite and affine type
95 H. L. Montgomery & R. C. Vaughan Multiplicative number theory, I
96 I. Chavel Riemannian geometry (2nd Edition)
97 A. Goldfeld Automorphic forms and L-functions for the group GL(n,R)
98 M. B. Marcus & J. Rosen Markov processes, Gaussian processes, and local times
99 P. Gille & T. Szamuely Central simple algebras and Galois cohomology
100 J. Bertoin Random fragmentation and coagulation processes
101 E. Frenkel Langlands correspondence for loop groups
102 A. Ambrosio & A. Malchiodi Nonlinear analysis and semilinear elliptic problems
103 T. Tao & V. H. Vu Additive combinatorics
104 E. B. Davies Linear operators and their spectra
105 K. Kodaira Complex analysis
106 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli Harmonic analysis on finite groups
107 H. Geiges An introduction to contact topology
108 J. Faraut Analysis on Lie groups: An Introduction
109 E. Park Complex topological K-theory
110 D. W. Stroock Partial differential equations for probabilists
111 A. Kirillov, Jr An introduction to Lie groups and Lie algebras
112 F. Gesztesy et al. Soliton equations and their algebro-geometric solutions, II
113 E. de Faria & W. de Melo Mathematical tools for one-dimensional dynamics
114 D. Applebaum Lévy processes and stochastic calculus (2nd Edition)
115 T. Szamuely Galois groups and fundamental groups
An Introduction to Random Matrices

GREG W. ANDERSON
University of Minnesota

ALICE GUIONNET
Ecole Normale Supérieure de Lyon

OFER ZEITOUNI
University of Minnesota and
Weizmann Institute of Science
To Meredith, Benoit and Naomi
# Contents

*Preface*  

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>xiii</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Real and complex Wigner matrices</td>
<td>6</td>
</tr>
<tr>
<td>2.1</td>
<td>Real Wigner matrices: traces, moments and combinatorics</td>
<td>6</td>
</tr>
<tr>
<td>2.1.1</td>
<td>The semicircle distribution, Catalan numbers and Dyck paths</td>
<td>7</td>
</tr>
<tr>
<td>2.1.2</td>
<td>Proof #1 of Wigner’s Theorem 2.1.1</td>
<td>10</td>
</tr>
<tr>
<td>2.1.3</td>
<td>Proof of Lemma 2.1.6: words and graphs</td>
<td>11</td>
</tr>
<tr>
<td>2.1.4</td>
<td>Proof of Lemma 2.1.7: sentences and graphs</td>
<td>17</td>
</tr>
<tr>
<td>2.1.5</td>
<td>Some useful approximations</td>
<td>21</td>
</tr>
<tr>
<td>2.1.6</td>
<td>Maximal eigenvalues and Füredi–Komlós enumeration</td>
<td>23</td>
</tr>
<tr>
<td>2.1.7</td>
<td>Central limit theorems for moments</td>
<td>29</td>
</tr>
<tr>
<td>2.2</td>
<td>Complex Wigner matrices</td>
<td>35</td>
</tr>
<tr>
<td>2.3</td>
<td>Concentration for functionals of random matrices and logarithmic Sobolev inequalities</td>
<td>38</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Smoothness properties of linear functions of the empirical measure</td>
<td>38</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Concentration inequalities for independent variables satisfying logarithmic Sobolev inequalities</td>
<td>39</td>
</tr>
<tr>
<td>2.3.3</td>
<td>Concentration for Wigner-type matrices</td>
<td>42</td>
</tr>
<tr>
<td>2.4</td>
<td>Stieltjes transforms and recursions</td>
<td>43</td>
</tr>
</tbody>
</table>
## CONTENTS

2.4.1 Gaussian Wigner matrices 45  
2.4.2 General Wigner matrices 47

2.5 Joint distribution of eigenvalues in the GOE and the GUE 50  
2.5.1 Definition and preliminary discussion of the GOE and the GUE 51  
2.5.2 Proof of the joint distribution of eigenvalues 54  
2.5.3 Selberg’s integral formula and proof of (2.5.4) 58  
2.5.4 Joint distribution of eigenvalues: alternative formulation 65  
2.5.5 Superposition and decimation relations 66

2.6 Large deviations for random matrices 70  
2.6.1 Large deviations for the empirical measure 71  
2.6.2 Large deviations for the top eigenvalue 81

2.7 Bibliographical notes 85

3 Hermite polynomials, spacings and limit distributions for the Gaussian ensembles 90  
3.1 Summary of main results: spacing distributions in the bulk and edge of the spectrum for the Gaussian ensembles 90  
3.1.1 Limit results for the GUE 90  
3.1.2 Generalizations: limit formulas for the GOE and GSE 93

3.2 Hermite polynomials and the GUE 94  
3.2.1 The GUE and determinantal laws 94  
3.2.2 Properties of the Hermite polynomials and oscillator wave-functions 99

3.3 The semicircle law revisited 101  
3.3.1 Calculation of moments of $L_N$ 102  
3.3.2 The Harer–Zagier recursion and Ledoux’s argument 103

3.4 Quick introduction to Fredholm determinants 107  
3.4.1 The setting, fundamental estimates and definition of the Fredholm determinant 107  
3.4.2 Definition of the Fredholm adjugant, Fredholm resolvent and a fundamental identity 110
3.5 Gap probabilities at 0 and proof of Theorem 3.1.1
3.5.1 The method of Laplace
3.5.2 Evaluation of the scaling limit: proof of Lemma 3.5.1
3.5.3 A complement: determinantal relations
3.6 Analysis of the sine-kernel
3.6.1 General differentiation formulas
3.6.2 Derivation of the differential equations: proof of Theorem 3.6.1
3.6.3 Reduction to Painlevé V
3.7 Edge-scaling: proof of Theorem 3.1.4
3.7.1 Vague convergence of the largest eigenvalue: proof of Theorem 3.1.4
3.7.2 Steepest descent: proof of Lemma 3.7.2
3.7.3 Properties of the Airy functions and proof of Lemma 3.7.1
3.8 Analysis of the Tracy–Widom distribution and proof of Theorem 3.1.5
3.8.1 The first standard moves of the game
3.8.2 The wrinkle in the carpet
3.8.3 Linkage to Painlevé II
3.9 Limiting behavior of the GOE and the GSE
3.9.1 Pfaffians and gap probabilities
3.9.2 Fredholm representation of gap probabilities
3.9.3 Limit calculations
3.9.4 Differential equations
3.10 Bibliographical notes

4 Some generalities
4.1 Joint distribution of eigenvalues in the classical matrix ensembles
4.1.1 Integration formulas for classical ensembles
4.1.2 Manifolds, volume measures and the coarea formula
CONTENTS

5.2.1 Algebraic noncommutative probability spaces and laws 325
5.2.2 $C^*$-probability spaces and the weak*-topology 329
5.2.3 $W^*$-probability spaces 339

5.3 Free independence 348
5.3.1 Independence and free independence 348
5.3.2 Free independence and combinatorics 354
5.3.3 Consequence of free independence: free convolution 359
5.3.4 Free central limit theorem 368
5.3.5 Freeness for unbounded variables 369

5.4 Link with random matrices 374
5.5 Convergence of the operator norm of polynomials of independent GUE matrices 394

5.6 Bibliographical notes 410

Appendices 414

A Linear algebra preliminaries 414
A.1 Identities and bounds 414
A.2 Perturbations for normal and Hermitian matrices 415
A.3 Noncommutative matrix $L^p$-norms 416
A.4 Brief review of resultants and discriminants 417

B Topological preliminaries 418
B.1 Generalities 418
B.2 Topological vector spaces and weak topologies 420
B.3 Banach and Polish spaces 422
B.4 Some elements of analysis 423

C Probability measures on Polish spaces 423
C.1 Generalities 423
C.2 Weak topology 425

D Basic notions of large deviations 427

E The skew field $H$ of quaternions and matrix theory over $F$ 430
E.1 Matrix terminology over $F$ and factorization theorems 431
### CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.2</td>
<td>The spectral theorem and key corollaries</td>
<td>433</td>
</tr>
<tr>
<td>E.3</td>
<td>A specialized result on projectors</td>
<td>434</td>
</tr>
<tr>
<td>E.4</td>
<td>Algebra for curvature computations</td>
<td>435</td>
</tr>
<tr>
<td>F</td>
<td>Manifolds</td>
<td>437</td>
</tr>
<tr>
<td>F.1</td>
<td>Manifolds embedded in Euclidean space</td>
<td>438</td>
</tr>
<tr>
<td>F.2</td>
<td>Proof of the coarea formula</td>
<td>442</td>
</tr>
<tr>
<td>F.3</td>
<td>Metrics, connections, curvature, Hessians, and the Laplace–Beltrami operator</td>
<td>445</td>
</tr>
<tr>
<td>G</td>
<td>Appendix on operator algebras</td>
<td>450</td>
</tr>
<tr>
<td>G.1</td>
<td>Basic definitions</td>
<td>450</td>
</tr>
<tr>
<td>G.2</td>
<td>Spectral properties</td>
<td>452</td>
</tr>
<tr>
<td>G.3</td>
<td>States and positivity</td>
<td>454</td>
</tr>
<tr>
<td>G.4</td>
<td>von Neumann algebras</td>
<td>455</td>
</tr>
<tr>
<td>G.5</td>
<td>Noncommutative functional calculus</td>
<td>457</td>
</tr>
<tr>
<td>H</td>
<td>Stochastic calculus notions</td>
<td>459</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>465</td>
</tr>
<tr>
<td>General conventions and notation</td>
<td></td>
<td>481</td>
</tr>
<tr>
<td>Index</td>
<td></td>
<td>484</td>
</tr>
</tbody>
</table>
Preface

The study of random matrices, and in particular the properties of their eigenvalues, has emerged from the applications, first in data analysis and later as statistical models for heavy-nuclei atoms. Thus, the field of random matrices owes its existence to applications. Over the years, however, it became clear that models related to random matrices play an important role in areas of pure mathematics. Moreover, the tools used in the study of random matrices came themselves from different and seemingly unrelated branches of mathematics.

At this point in time, the topic has evolved enough that the newcomer, especially if coming from the field of probability theory, faces a formidable and somewhat confusing task in trying to access the research literature. Furthermore, the background expected of such a newcomer is diverse, and often has to be supplemented before a serious study of random matrices can begin.

We believe that many parts of the field of random matrices are now developed enough to enable one to expose the basic ideas in a systematic and coherent way. Indeed, such a treatise, geared toward theoretical physicists, has existed for some time, in the form of Mehta’s superb book [Meh91]. Our goal in writing this book has been to present a rigorous introduction to the basic theory of random matrices, including free probability, that is sufficiently self-contained to be accessible to graduate students in mathematics or related sciences who have mastered probability theory at the graduate level, but have not necessarily been exposed to advanced notions of functional analysis, algebra or geometry. Along the way, enough techniques are introduced that we hope will allow readers to continue their journey into the current research literature.

This project started as notes for a class on random matrices that two of us (G. A. and O. Z.) taught in the University of Minnesota in the fall of 2003, and notes for a course in the probability summer school in St. Flour taught by A. G. in the
The comments of participants in these courses, and in particular A. Bandyopadhyay, H. Dong, K. Hoffman-Credner, A. Klenke, D. Stanton and P.M. Zamfir, were extremely useful. As these notes evolved, we taught from them again at the University of Minnesota, the University of California at Berkeley, the Technion and the Weizmann Institute, and received more much appreciated feedback from the participants in those courses. Finally, when expanding and refining these course notes, we have profited from the comments and questions of many colleagues. We would like in particular to thank G. Ben Arous, F. Benaych-Georges, P. Biane, P. Deift, A. Dembo, P. Diaconis, U. Haagerup, V. Jones, M. Krishnapur, Y. Peres, R. Pinsky, G. Pisier, B. Rider, D. Shlyakhtenko, B. Solel, A. Soshnikov, R. Speicher, T. Suidan, C. Tracy, B. Virag and D. Voiculescu for their suggestions, corrections and patience in answering our questions or explaining their work to us. Of course, any remaining mistakes and unclear passages are fully our responsibility.

GREG ANDERSON
ALICE GUIONNET
OFER ZEITOUNI

MINNEAPOLIS, MINNESOTA
LYON, FRANCE
REHOVOT, ISRAEL