DIFFERENTIAL EQUATIONS FOR ENGINEERS

This book presents a systematic and comprehensive introduction to ordinary differential equations for engineering students and practitioners. Mathematical concepts and various techniques are presented in a clear, logical, and concise manner. Various visual features are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. Readers are motivated by a focus on the relevance of differential equations through their applications in various engineering disciplines. Studies of various types of differential equations are determined by engineering applications. Theory and techniques for solving differential equations are then applied to solve practical engineering problems. Detailed step-by-step analysis is presented to model the engineering problems using differential equations from physical principles and to solve the differential equations using the easiest possible method. Such a detailed, step-by-step approach, especially when applied to practical engineering problems, helps the readers to develop problem-solving skills.

This book is suitable for use not only as a textbook on ordinary differential equations for undergraduate students in an engineering program but also as a guide to self-study. It can also be used as a reference after students have completed learning the subject.

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Differential Equations for Engineers

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Preface

Background

Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits, are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

I have been teaching differential equations to engineering students for the past two decades. Most, if not all, of the textbooks are written by mathematicians with little engineering background. Based on my experience and feedback from students, the following lists some of the gaps frequently seen in current textbooks:

- A major focus is put on explaining mathematical concepts

  For engineers, the purpose of learning the theory of differential equations is to be able to solve practical problems where differential equations are used. For engineering students, it is more important to know the applications and techniques for solving application problems than to delve into the nuances of mathematical concepts and theorems. Knowing the appropriate applications can motivate them to study the mathematical concepts and techniques. However, it is much more challenging to model an application problem using physical principles and then solve the resulting differential equations than it is to merely carry out mathematical exercises.

- Insufficient emphasis is placed on the step-by-step problem solving techniques

  Engineering students do not usually have the same mathematical background and interest as students who major in mathematics. Mathematicians are more interested if: (1) there are solutions to a differential equation or a system of differential equations; (2) the solutions are unique under a certain set of conditions; and (3) the differential equations can be solved. On the other hand,

engineers are more interested in mathematical modeling of a practical problem and actually solving the equations to find the solutions using the easiest possible method. Hence, a detailed step-by-step approach, especially applied to practical engineering problems, helps students to develop problem solving skills.

- Presentations are usually formula-driven with little variation in visual design

  It is very difficult to attract students to read boring formulas without variation of presentation. Readers often miss the points of importance.

Objectives

This book addresses the needs of engineering students and aims to achieve the following objectives:

- To motivate students on the relevance of differential equations in engineering through their applications in various engineering disciplines. Studies of various types of differential equations are motivated by engineering applications; theory and techniques for solving differential equations are then applied to solve practical engineering problems.
- To have a balance between theory and applications. This book could be used as a reference after students have completed learning the subject. As a reference, it has to be reasonably comprehensive and complete. Detailed step-by-step analysis is presented to model the engineering problems using differential equations and to solve the differential equations.
- To present the mathematical concepts and various techniques in a clear, logical and concise manner. Various visual features, such as side-notes (preceded by the  symbol), different fonts and shades, are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. This book is not only suitable as a textbook for classroom use but also is easy for self-study. As a textbook, it has to be easy to understand. For self-study, the presentation is detailed with all necessary steps and useful formulas given as side-notes.

Scope

This book is primarily for engineering students and practitioners as the main audience. It is suitable as a textbook on ordinary differential equations for undergraduate students in an engineering program. Such a course is usually offered in the second year after students have taken calculus and linear algebra in the first year. Although it is assumed that students have a working knowledge of calculus and linear algebra, some important concepts and results are reviewed when they are first used so as to refresh their memory.

Chapter 1 first presents some motivating examples, which will be studied in detail later in the book, to illustrate how differential equations arise in engineering applications. Some basic general concepts of differential equations are then introduced.

In Chapter 2, various techniques for solving first-order and simple higher-order ordinary differential equations are presented. These methods are then applied in Chapter 3 to study various application problems involving first-order and simple higher-order differential equations.

Chapter 4 studies linear ordinary differential equations. Complementary solutions are obtained through the characteristic equations and characteristic numbers. Particular solutions are obtained using the method of undetermined coefficients, the operator method, and the method of variation of parameters. Applications involving linear ordinary differential equations are presented in Chapter 5.

Solutions of linear ordinary differential equations using the Laplace transform are studied in Chapter 6, emphasizing functions involving Heaviside step function and Dirac delta function.

Chapter 7 studies solutions of systems of linear ordinary differential equations. The method of operator, the method of Laplace transform, and the matrix method are introduced. Applications involving systems of linear ordinary differential equations are considered in Chapter 8.

In Chapter 9, solutions of ordinary differential equations in series about an ordinary point and a regular singular point are presented. Applications of Bessel's equation in engineering are considered.

Some classical methods, including forward and backward Euler method, improved Euler method, and Runge-Kutta methods, are presented in Chapter 10 for numerical solutions of ordinary differential equations.

In Chapter 11, the method of separation of variables is applied to solve partial differential equations. When the method is applicable, it converts a partial differential equation into a set of ordinary differential equations. Flexural vibration of beams and heat conduction are studied as examples of application.

Solutions of ordinary differential equations using *Maple* are presented in Chapter 12. Symbolic computation software, such as *Maple*, is very efficient in solving problems involving ordinary differential equations. However, it cannot replace learning and thinking, especially mathematical modeling. It is important to develop analytical skills and proficiency through "hand" calculations, as has been done in previous chapters. This will also help the development of insight into the problems and appreciation of the solution process. For this reason, solutions of ordinary differential equations using *Maple* is presented in the last chapter of the book instead of a scattering throughout the book.

The book covers a wide range of materials on ordinary differential equations and their engineering applications. There are more than enough materials for a one-term (semester) undergraduate course. Instructors can select the materials according to the curriculum. Drafts of this book were used as the textbook in a one-term undergraduate course at the University of Waterloo.

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First and foremost, my sincere appreciation goes to my students. It is the students who give me a stage where I can cultivate my talent and passion for teaching. It is for the students that this book is written, as my small contribution to their success in academic and professional careers. My undergraduate students who have used the draft of this book as a textbook have made many encouraging comments and constructive suggestions.

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I appreciate hearing your comments through email (xie@uwaterloo.ca) or regular correspondence.

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