CHAPTER 1

Introduction

1.1 Motivating Examples

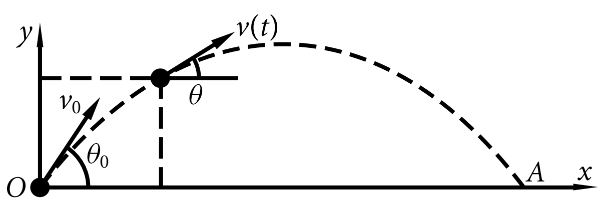
Differential equations have wide applications in various engineering and science disciplines. In general, modeling variations of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, or concentration of a pollutant, with the change of time t or location, such as the coordinates (x, y, z), or both would require differential equations. Similarly, studying the variation of a physical quantity on other physical quantities would lead to differential equations. For example, the change of strain on stress for some viscoelastic materials follows a differential equation.

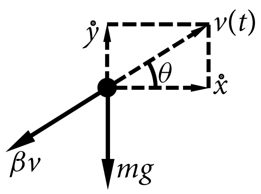
It is important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

In this section, a few examples are presented to illustrate how practical problems are modeled mathematically and how differential equations arise in them.

Motivating Example 1

First consider the projectile of a mass m launched with initial velocity v_0 at angle θ_0 at time $t=0$, as shown.





The atmosphere exerts a resistance force on the mass, which is proportional to the instantaneous velocity of the mass, i.e., $R=\beta v$, where β is a constant, and is opposite to the direction of the velocity of the mass. Set up the Cartesian coordinate system as shown by placing the origin at the point from where the mass m is launched.

At time t, the mass is at location $\big(x(t), y(t)\big)$. The instantaneous velocity of the mass in the x- and y-directions are $\dot{x}(t)$ and $\dot{y}(t)$, respectively. Hence the velocity of the mass is $v(t)=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}$ at the angle $\theta(t)=\tan^{-1}\big[\dot{y}(t)/\dot{x}(t)\big]$.

The mass is subjected to two forces: the vertical downward gravity mg and the resistance force $R(t)=\beta v(t)$.

The equations of motion of the mass can be established using Newton's Second Law: $F=\sum ma$. The x-component of the resistance force is $-R(t)\cos\theta(t)$. In the y-direction, the component of the resistance force is $-R(t)\sin\theta(t)$. Hence, applying Newton's Second Law yields

$$x\text{-direction:}\quad ma_x=\sum F_x \quad\Longrightarrow\quad m\ddot{x}(t)=-R(t)\cos\theta(t),$$

$$y\text{-direction:}\quad ma_y=\sum F_y \quad\Longrightarrow\quad m\ddot{y}(t)=-mg-R(t)\sin\theta(t).$$

Since

$$\theta(t)=\tan^{-1}\frac{\dot{y}(t)}{\dot{x}(t)} \quad\Longrightarrow\quad \cos\theta=\frac{\dot{x}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}},\quad \sin\theta=\frac{\dot{y}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}},$$

the equations of motion become

$$m\ddot{x}(t)=-\beta v(t)\cdot\frac{\dot{x}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}} \quad\Longrightarrow\quad m\ddot{x}(t)+\beta\dot{x}(t)=0,$$

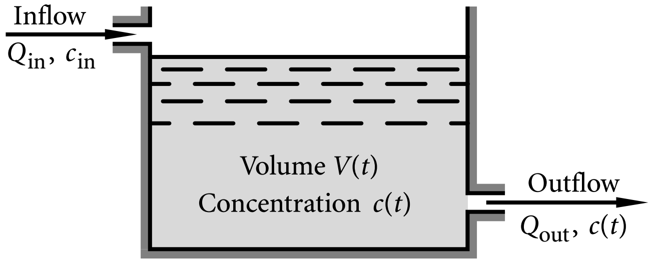
$$m\ddot{y}(t)=-mg-\beta v(t)\cdot\frac{\dot{y}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}} \quad\Longrightarrow\quad m\ddot{y}(t)+\beta\dot{y}(t)=-mg,$$

in which the initial conditions are at time $t=0$: $x(0)=0$, $y(0)=0$, $\dot{x}(0)=v_0\cos\theta_0$, $\dot{y}(0)=v_0\sin\theta_0$. The equations of motion are two equations involving the first- and second-order derivatives $\dot{x}(t)$, $\dot{y}(t)$, $\ddot{x}(t)$, and $\ddot{y}(t)$. These equations are called, as will be defined later, a system of two second-order ordinary differential equations.

Because of the complexity of the problems, in the following examples, the problems are described and the governing equations are presented without detailed derivation. These problems will be investigated in details in later chapters when applications of various types of differential equations are studied.

Motivating Example 2

A tank contains a liquid of volume $V(t)$, which is polluted with a pollutant concentration in *percentage* of $c(t)$ at time t. To reduce the pollutant concentration, an inflow of rate Q_{in} is injected to the tank. Unfortunately, the inflow is also polluted but to a lesser degree with a pollutant concentration c_{in}. It is assumed that the inflow is perfectly mixed with the liquid in the tank instantaneously. An outflow of rate Q_{out} is removed from the tank as shown. Suppose that, at time $t=0$, the volume of the liquid is V_0 with a pollutant concentration of c_0.

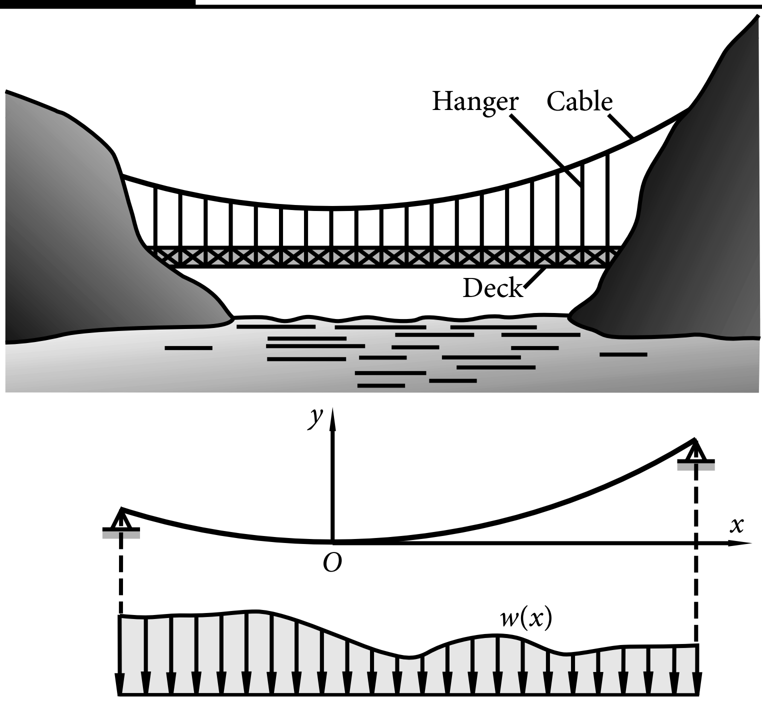


The equation governing the pollutant concentration $c(t)$ is given by

$$\big[V_0 + (Q_{\text{in}} - Q_{\text{out}})t\big]\frac{\mathrm{d}c(t)}{\mathrm{d}t} + Q_{\text{in}}c(t) = Q_{\text{in}}c_{\text{in}},$$

with initial condition $c(0)=c_0$. This is a first-order ordinary differential equation.

Motivating Example 3



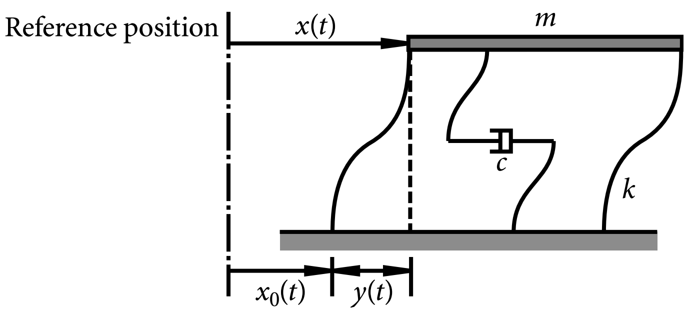
Consider the suspension bridge as shown, which consists of the main cable, the hangers, and the deck. The self-weight of the deck and the loads applied on the deck are transferred to the cable through the hangers.

Set up the Cartesian coordinate system by placing the origin O at the lowest point of the cable. The cable can be modeled as subjected to a distributed load $w(x)$. The equation governing the shape of the cable is given by

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H},$$

where H is the tension in the cable at the lowest point O. This is a second-order ordinary differential equation.

Motivating Example 4



Consider the vibration of a single-story shear building under the excitation of earthquake. The shear building consists of a rigid girder of mass m supported by columns of combined stiffness k. The vibration of the girder can be described by the horizontal displacement $x(t)$. The earthquake is modeled by the displacement of the ground $x_0(t)$ as shown. When the girder vibrates, there is a damping force due to the internal friction between various components of the building, given by $c\big[\dot{x}(t)-\dot{x}_0(t)\big]$, where c is the damping coefficient.

The relative displacement $y(t)=x(t)-x_0(t)$ between the girder and the ground is governed by the equation

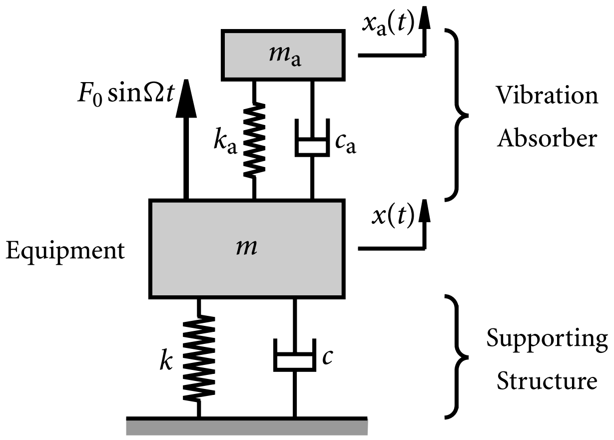
$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -m\ddot{x}_0(t),$$

which is a second-order linear ordinary differential equation.

Motivating Example 5

In many engineering applications, an equipment of mass m is usually mounted on a supporting structure that can be modeled as a spring of stiffness k and a damper of damping coefficient c as shown in the following figure. Due to unbalanced mass in rotating components or other excitation mechanisms, the equipment is subjected to a harmonic force $F_0 \sin \Omega t$. The vibration of the mass is described by the vertical displacement $x(t)$. When the excitation frequency Ω is close to $\omega_0=\sqrt{k/m}$, which is the natural circular frequency of the equipment and its support, vibration of large amplitudes occurs.

In order to reduce the vibration of the equipment, a vibration absorber is mounted on the equipment. The vibration absorber can be modeled as a mass m_a, a spring of stiffness k_a, and a damper of damping coefficient c_a. The vibration of the absorber is described by the vertical displacement $x_a(t)$.



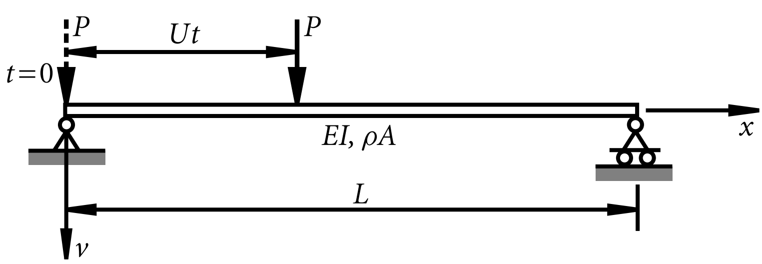
The equations of motion governing the vibration of the equipment and the absorber are given by

$$m\ddot{x} + (c+c_a)\dot{x} + (k+k_a)x - c_a\dot{x}_a - k_a x_a = F_0 \sin \Omega t,$$

$$m_a\ddot{x}_a + c_a\dot{x}_a + k_a x_a - c_a\dot{x} - k_a x = 0,$$

which comprises a system of two coupled second-order linear ordinary differential equations.

Motivating Example 6



A bridge may be modeled as a simply supported beam of length L, mass density per unit length ρA, and flexural rigidity EI as shown. A vehicle of weight P crosses the bridge at a constant speed U. Suppose at time $t=0$, the vehicle is at the left end of the bridge and the bridge is at rest. The deflection of the bridge is $v(x,t)$, which is a function of both location x and time t. The equation governing $v(x,t)$ is the partial differential equation

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} = P\,\delta(x-Ut),$$

where $\delta(x-a)$ is the Dirac delta function. The equation of motion satisfies the initial conditions

$$v(x,0)=0, \qquad \frac{\partial v(x,t)}{\partial t}\bigg|_{t=0}=0,$$

and the boundary conditions

$$v(0,t)=v(L,t)=0, \qquad \frac{\partial^2 v(x,t)}{\partial x^2}\bigg|_{x=0}=\frac{\partial^2 v(x,t)}{\partial x^2}\bigg|_{x=L}=0.$$

1.2 General Concepts and Definitions

In this section, some general concepts and definitions of ordinary and partial differential equations are presented.

Let x be an independent variable and y be a dependent variable. An equation that involves x, y and various derivatives of y is called a *differential equation* (DE). For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2y+\sin x, \qquad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\mathrm{e}^x+2=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

are differential equations.

Definition — Ordinary Differential Equation

In general, an equation of the form

$$F\left(x,\ y,\ \frac{\mathrm{d}y}{\mathrm{d}x},\ \dots,\ \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right)=0$$

is an *Ordinary Differential Equation* (ODE).

It is called an *ordinary* differential equation because there is only one independent variable and only ordinary derivatives (not partial derivatives) are involved.

Definition — Order of a Differential Equation

The *order* of a differential equation is the order of the highest derivative appearing in the differential equation.

Definition — Linear and Nonlinear Differential Equations

If y and its various derivatives y', y'', ... appear linearly in the equation, it is a *linear* differential equation; otherwise, it is *nonlinear*.

For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\omega^2 y=\sin x, \quad \omega=\text{constant},$$ ✍ Second-order, linear

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+4y=\cos x,$$ ✍ First-order, nonlinear because of the term $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$

$$x^3 \frac{d^3y}{dx^3} + 5x \frac{dy}{dx} + 6y = e^x,$$ ✍ Third-order, linear

$$\frac{d^2y}{dx^2} + y \frac{dy}{dx} + 2y = x.$$ ✍ Second-order, nonlinear because of the term $y \dfrac{dy}{dx}$

Sometimes, the roles of independent and dependent variables can be exchanged to render a differential equation linear. For example,

$$\frac{d^2x}{dy^2} - x\sqrt{y} = 5$$

is a second-order linear equation with y being regarded as the independent variable and x the dependent variable.

In some applications, the roles of independent and dependent variables are obvious. For example, in a differential equation governing the variation of temperature T with time t, the time variable t is the independent variable and the temperature T is the dependent variable; time t cannot be the dependent variable. In other applications, the roles of independent and dependent variables are interchangeable. For example, in a differential equation governing the relationship between temperature T and pressure p, the temperature T can be considered as the independent variable and the pressure p the dependent variable, or vice versa.

Definition — Linear Ordinary Differential Equations

The general form of an nth-order linear ordinary differential equation is

$$a_n(x) \frac{d^ny}{dx^n} + a_{n-1}(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x).$$

If $a_0(x), a_1(x), \ldots, a_n(x)$ are constants, the ordinary differential equation is said to have *constant coefficients*; otherwise it is said to have *variable coefficients*.

For example,

$$\frac{d^2y}{dx^2} + 0.1 \frac{dy}{dx} + 4y = 10\cos 2x,$$ ✍ Second-order linear, constant coefficients

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0, \quad x > 0, \quad \nu \geqslant 0 \text{ is a constant.}$$

✍ Second-order linear, variable coefficients (Bessel's equation)

Definition — Homogeneous and Nonhomogeneous Differential Equations

A differential equation is said to be *homogeneous* if it has zero as a solution; otherwise, it is *nonhomogeneous*.

For example,

$$\frac{d^2y}{dx^2} + 0.1\frac{dy}{dx} + 4y = 0, \qquad \text{✍ Homogeneous}$$

$$\frac{d^2y}{dx^2} + 0.1\frac{dy}{dx} + 4y = 2\sin 2x + 5\cos 3x. \qquad \text{✍ Nonhomogeneous}$$

Note that a homogeneous differential equation may have distinctively different meanings in different situations (see Section 2.2).

Partial Differential Equations

Definition — Partial Differential Equations

If the dependent variable u is a function of more than one independent variable, say $x_1, x_2, \ldots, x_m$, an equation involving the variables $x_1, x_2, \ldots, x_m$, u and various partial derivatives of u with respect to $x_1, x_2, \ldots, x_m$ is called a *Partial Differential Equation* (PDE).

For example,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}, \quad \alpha = \text{constant}, \qquad \text{✍ Heat equation in one-dimension}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad \text{✍ Poisson's equation in two-dimensions; Laplace's equation if } f(x, y) = 0$$

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \qquad \text{✍ Biharmonic equation in two-dimensions}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad \text{✍ Heat equation in three-dimensions}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{✍ Laplace's equation in three-dimensions}$$

General and Particular Solutions

Definition — Solution of a Differential Equation

For an nth-order ordinary differential equation $F\big(x, y, y', \ldots, y^{(n)}\big) = 0$, a function $y = y(x)$, which is n times differentiable and satisfies the differential equation in some interval $a < x < b$ when substituted into the equation, is called a *solution* of the differential equation over the interval $a < x < b$.

Consider the first-order differential equation

$$\frac{dy}{dx} = 3.$$

Integrating with respect to x yields the general solution

$$y = 3x + C, \qquad C = \text{constant}.$$

The general solution of the differential equation, which includes all possible solutions, is a family of straight lines with slope equal to 3. On the other hand, $y = 3x$ is a particular solution passing through the origin, with the constant C being 0.

Consider the differential equation

$$\frac{d^3y}{dx^3} = 48x.$$

Integrating both sides of the equation with respect to x gives

$$\frac{d^2y}{dx^2} = 24x^2 + C_1.$$

Integrating with respect to x again yields

$$\frac{dy}{dx} = 8x^3 + C_1 x + C_2.$$

Integrating with respect to x once more results in the general solution

$$y = 2x^4 + \tfrac{1}{2} C_1 x^2 + C_2 x + C_3,$$

where C_1, C_2, C_3 are arbitrary constants. When the constants C_1, C_2, C_3 take specific values, one obtains particular solutions. For example,

$$\begin{aligned} y &= 2x^4 + 3x^2 + 1, && C_1 = 6, \quad C_2 = 0, \quad C_3 = 1, \\ y &= 2x^4 + x^2 + 3x + 5, && C_1 = 2, \quad C_2 = 3, \quad C_3 = 5, \end{aligned}$$

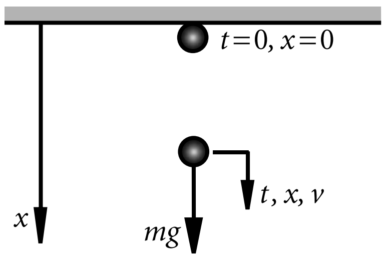
are two particular solutions.

Remarks: In general, an nth-order ordinary differential equation will contain n arbitrary constants in its general solution. Hence, for an nth-order ordinary differential equation, n conditions are required to determine the n constants to yield a particular solution.

In applications, there are usually two types of conditions that can be used to determine the constants.

Illustrative Example

Consider the motion of an object dropped vertically at time $t = 0$ from $x = 0$ as shown in the following figure. Suppose that there is no resistance from the medium.



The equation of motion is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g,$$

and the general solution is, by integrating both sides of the equation with respect to t twice,

$$x(t) = C_0 + C_1 t + \tfrac{1}{2}gt^2.$$

The following are two possible ways of specifying the conditions.

Initial Value Problem

If the object is dropped with initial velocity v_0, the conditions required are

$$\text{at time } t=0: \quad x(0)=0, \quad \dot{x}(0) = \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = v_0.$$

The constants C_0 and C_1 can be determined from these two conditions and the solution of the differential equation is

$$x(t) = v_0 t + \tfrac{1}{2}gt^2.$$

In this case, the differential equation is required to satisfy conditions specified at one value of t, i.e., $t=0$.

Definition — Initial Value Problem

If a differential equation is required to satisfy conditions on the dependent variable and its derivatives specified at *one value* of the independent variable, these conditions are called *initial conditions* and the problem is called an *initial value problem*.

Boundary Value Problem

If the object is required to reach $x=L$ at time $t=T$, $L \geqslant \frac{1}{2}gT^2$, the conditions can be specified as

$$\text{at time } t=0: \quad x(0)=0; \qquad \text{at time } t=T: \quad x(T)=L.$$