Complex-Valued Matrix Derivatives

In this complete introduction to the theory of finding derivatives of scalar-, vector-, and matrix-valued functions in relation to complex matrix variables, Hjørungnes describes an essential set of mathematical tools for solving research problems where unknown parameters are contained in complex-valued matrices. Self-contained and easy to follow, this singular reference uses numerous practical examples from signal processing and communications to demonstrate how these tools can be used to analyze and optimize the performance of engineering systems. This is the first book on complex-valued matrix derivatives from an engineering perspective. It covers both unpatterned and patterned matrices, uses the latest research examples to illustrate concepts, and includes applications in a range of areas, such as wireless communications, control theory, adaptive filtering, resource management, and digital signal processing. The book includes eighty-one end-of-chapter exercises and a complete solutions manual (available on the Web).

Are Hjørungnes is a Professor in the Faculty of Mathematics and Natural Sciences at the University of Oslo, Norway. He is an Editor of the *IEEE Transactions on Wireless Communications*, and has served as a Guest Editor of the *IEEE Journal of Selected Topics in Signal Processing* and the *IEEE Journal on Selected Areas in Communications.*

This book addresses the problem of complex-valued derivatives in a wide range of contexts. The mathematical presentation is rigorous but its structured and comprehensive presentation makes the information easily accessible. Clearly, it is an invaluable reference to researchers, professionals and students dealing with functions of complex-valued matrices that arise frequently in many different areas. Throughout the book the examples and exercises help the reader learn how to apply the results presented in the propositions, lemmas and theorems. In conclusion, this book provides a well organized, easy to read, authoritative and unique presentation that everyone looking to exploit complex functions should have available in their own shelves and libraries.

Professor Paulo S. R. Diniz, Federal University of Rio de Janeiro

Complex vector and matrix optimization problems are often encountered by researchers in the electrical engineering fields and much beyond. Their solution, which can sometimes be reached from using existing standard algebra literature, may however be a time consuming and sometimes difficult process. This is particularly so when complicated cost function and constraint expressions arise. This book brings together several mathematical theories in a novel manner to offer a beautifully unified and systematic methodology for approaching such problems. It will no doubt be a great companion to many researchers and engineers alike.

Professor David Gesbert, EURECOM, Sophia-Antipolis, France

Complex-Valued Matrix Derivatives

With Applications in Signal Processing and Communications

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521192644

First published 2011

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication data
Hjørungnes, Are.
Complex-Valued Matrix Derivatives : With Applications in Signal Processing and Communications /
Are Hjørungnes.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-521-19264-4 (hardback)
1. Matrix derivatives. 2. Systems engineering. 3. Signal processing – Mathematical models.
4. Telecommunication – Mathematical models. I. Title.
TA347.D4H56 2011
621.382′2 – dc22 2010046598

ISBN 978-0-521-19264-4 Hardback

Additional resources for this publication at www.cambridge.org/hjorungnes

To my parents, Tove and Odd

Cambridge University Press
978-0-521-19264-4 - Complex-Valued Matrix Derivatives: With Applications in Signal Processing and Communications
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Preface

This book is written as an engineering-oriented mathematics book. It introduces the field involved in finding derivatives of complex-valued functions with respect to complex-valued matrices, in which the output of the function may be a scalar, a vector, or a matrix. The theory of complex-valued matrix derivatives, collected in this book, will benefit researchers and engineers working in fields such as signal processing and communications. Theories for finding complex-valued derivatives with respect to both complex-valued matrices with independent components and matrices that have certain dependencies among the components are developed and illustrative examples that show how to find such derivatives are presented. Key results are summarized in tables. Through several research-related examples, it will be shown how complex-valued matrix derivatives can be used as a tool to solve research problems in the fields of signal processing and communications.

This book is suitable for M.S. and Ph.D. students, researchers, engineers, and professors working in signal processing, communications, and other fields in which the unknown variables of a problem can be expressed as complex-valued matrices. The goal of the book is to present the tools of complex-valued matrix derivatives such that the reader is able to use these theories to solve open research problems in his or her own field. Depending on the nature of the problem, the components inside the unknown matrix might be independent, or certain interrelations might exist among the components. Matrices with independent components are called *unpatterned* and, if functional dependencies exist among the elements, the matrix is called *patterned* or *structured*. Derivatives relating to complex matrices with independent components are called *complex-valued matrix derivatives*; derivatives relating to matrices that belong to sets that may contain certain structures are called *generalized complex-valued matrix derivatives*. Researchers and engineers can use the theories presented in this book to optimize systems that contain complex-valued matrices. The theories in this book can be used as tools for solving problems, with the aim of minimizing or maximizing real-valued objective functions with respect to complex-valued matrices. People who work in research and development for future signal processing and communication systems can benefit from this book because they can use the presented material to optimize their complex-valued design parameters.

Book Overview

This book contains seven chapters. Chapter 1 gives a short introduction to the book. Mathematical background material needed throughout the book is presented in Chapter 2. Complex differentials and the definition of complex-valued derivatives are provided in Chapter 3, and, in addition, several important theorems are proved. Chapter 4 uses many examples to show the reader how complex-valued derivatives can be found for nine types of functions, depending on function output (scalar, vector, or matrix) and input parameters (scalar, vector, or matrix). Second-order derivatives are presented in Chapter 5, which shows how to find the Hessian matrices of complex-valued scalar, vector, and matrix functions for unpatterned matrix input variables. Chapter 6 is devoted to the theory of generalized complex-valued matrix derivatives. This theory includes derivatives with respect to complex-valued matrices that belong to certain sets, such as Hermitian matrices. Chapter 7 presents several examples that show how the theory can be used as an important tool to solve research problems related to signal processing and communications. All chapters except Chapter 1 include at least 11 exercises with relevant problems taken from the chapters. A solution manual that provides complete solutions to problems in all exercises is available at www.cambridge.org/hjorungnes.

I will be very interested to hear from you, the reader, on any comments or suggestions regarding this book.

Acknowledgments

During my Ph.D. studies, I started to work in the field of complex-valued matrix derivatives. I am very grateful to my Ph.D. advisor Professor Tor A. Ramstad at the Norwegian University of Science and Technology for everything he has taught me and, in particular, for leading me onto the path to matrix derivatives. My work on matrix derivatives was intensified when I worked as a postdoctoral Research Fellow at Helsinki University of Technology and the University of Oslo. The idea of writing a book developed gradually, but actual work on it started at the beginning of 2008.

I would like to thank the people at Cambridge University Press for their help. I would especially like to thank Dr. Phil Meyler for the opportunity to publish this book with Cambridge and Sarah Finlay, Cambridge Publishing Assistant, for her help with its practical concerns during this preparation. Thanks also go to the reviewers of my book proposal for helping me improve my work.

I would like to acknowledge the financial support of the Research Council of Norway for its funding of the FRITEK project “Theoretical Foundations of Mobile Flexible Networks – THEFONE” (project number 197565/V30). The THEFONE project contains one work package on complex-valued matrix derivatives.

I am grateful to Professor Zhu Han of the University of Houston for discussing with me points on book writing and book proposals, especially during my visit to the University of Houston in December, 2008. I thank Professor Paulo S. R. Diniz of the Federal University of Rio de Janeiro for helping me with questions about book proposals and other matters relating to book writing. I am grateful to Professor David Gesbert of EURECOM and Professor Daniel P. Palomar of Hong Kong University of Science and Technology for their help in organizing some parts of this book and for their valuable feedback and suggestions during its early stages. Thanks also go to Professor Visa Koivunen of the Aalto University School of Science and Technology for encouraging me to collect material on complex-valued matrix derivatives and for their valuable comments on how to organize the material. I thank Professor Kenneth Kreutz-Delgado for interesting discussions during my visit to the University of California, San Diego, in December, 2009, and pointing out several relevant references. Dr. Per Christian Moan helped by discussing several topics in this book in an inspiring and friendly atmosphere. I am grateful to Professor Hans Brodersen and Professor John Rognes, both of the University of Oslo, for discussions related to the initial material on manifold. I also thank Professor Aleksandar Kavčić of the University of Hawaiʻi at Mānoa for helping arrange my sabbatical in Hawaiʻi from mid-July, 2010, to mid-July, 2011.

Thanks go to the postdoctoral research fellows and Ph.D. students in my research group, in addition to all the inspiring guests who visited with my group while I was writing this book. Several people have helped me find errors and improve the material. I would especially like to thank Dr. Ninoslav Marina, who has been of great help in finding typographical errors. I thank Professor Manav R. Bhatnagar of the Indian Institute of Technology Delhi; Professor Dusit Niyato of Nanyang Technological University; Dr. Xiangyun Zhou; and Dr. David K. Choi for their suggestions. In addition, thanks go to Martin Makundi, Dr. Marius Sîrbu, Dr. Timo Roman, and Dr. Traian Abrudan, who made corrections on early versions of this book.

Finally, I thank my friends and family for their support during the preparation and writing of this book.

Abbreviations

BER	bit error rate
CDMA	code division multiple access
CFO	carrier frequency offset
DFT	discrete Fourier transform
FIR	finite impulse response
i.i.d.	independent and identically distributed
LOS	line-of-sight
LTI	linear time-invariant
MIMO	multiple-input multiple-output
MLD	maximum likelihood decoding
MSE	mean square error
OFDM	orthogonal frequency-division multiplexing
OSTBC	orthogonal space-time block code
PAM	pulse amplitude modulation
PSK	phase shift keying
QAM	quadrature amplitude modulation
SER	symbol error rate
SISO	single-input single-output
SNR	signal-to-noise ratio
SVD	singular value decomposition
TDMA	time division multiple access
wrt.	with respect to

Nomenclature

$\otimes$	Kronecker product
$\odot$	Hadamard product
$\triangleq$	defined equal to
$\subseteq$	subset of
$\subset$	proper subset of
$\wedge$	logical conjunction
$\forall$	for all
$\sum$	summation
$\prod$	product
$\times$	Cartesian product
$\int$	integral
$\leq$	less than or equal to
$<$	strictly less than
$\geq$	greater than or equal to
$>$	strictly greater than
$\succeq$	$\boldsymbol{S} \succeq \mathbf{0}_{N \times N}$ means that $\boldsymbol{S}$ is positive semidefinite
∞	infinity
$\neq$	not equal to
$\|$	such that
$\|\cdot\|$	(1) $\|z\| \geq 0$ returns the absolute value of the number $z \in \mathbb{C}$ (2) $\|\boldsymbol{z}\| \in (\mathbb{R}^+ \cup \{0\})^{N \times 1}$ returns the component-wise absolute values of the vector $\boldsymbol{z} \in \mathbb{C}^{N \times 1}$ (3) $\|\mathcal{A}\|$ returns the cardinality of the set $\mathcal{A}$
$\angle(\cdot)$	(1) $\angle z$ returns the principal value of the argument of the complex input variable z (2) $\angle \boldsymbol{z} \in (-\pi, \pi]^{N \times 1}$ returns the component-wise principal argument of the vector $\boldsymbol{z} \in \mathbb{C}^{N \times 1}$
$\sim$	is statistically distributed according to
$\mathbf{0}_{M \times N}$	$M \times N$ matrix containing only zeros
$\mathbf{1}_{M \times N}$	$M \times N$ matrix containing only ones
$(\cdot)^*$	$\boldsymbol{Z}^*$ means component-wise complex conjugation of the elements in the matrix $\boldsymbol{Z}$
$\emptyset$	empty set
$\setminus$	set difference

$(\cdot)^{-1}$	matrix inverse
$\lvert\cdot\rvert^{-1}$	if $\boldsymbol{z} \in \{\mathbb{C} \setminus \{0\}\}^{N\times 1}$, then $\lvert\boldsymbol{z}\rvert^{-1}$ returns a vector in $(\mathbb{R}^{+})^{N\times 1}$ with the inverse of the component-wise absolute values of $\boldsymbol{z}$
$(\cdot)^{+}$	Moore-Penrose inverse
$(\cdot)^{\#}$	adjoint of a matrix
$\mathbb{C}$	set of complex numbers
$\mathcal{C}(\boldsymbol{A})$	column space of the matrix $\boldsymbol{A}$
$\mathcal{CN}$	complex normally distributed
$\mathcal{N}(\boldsymbol{A})$	null space of the matrix $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	row space of the matrix $\boldsymbol{A}$
$\delta_{i,j}$	Kronecker delta function with two input arguments
$\delta_{i,j,k}$	Kronecker delta function with three input arguments
$\lambda_{\max}(\cdot)$	maximum eigenvalue of the input matrix, which must be Hermitian
$\lambda_{\min}(\cdot)$	minimum eigenvalue of the input matrix, which must be Hermitian
μ	Lagrange multiplier
$\nabla_{\boldsymbol{Z}} f$	the gradient of f with respect to $\boldsymbol{Z}^*$ and $\nabla_{\boldsymbol{Z}} f \in \mathbb{C}^{N\times Q}$ when $\boldsymbol{Z} \in \mathbb{C}^{N\times Q}$
$\frac{\partial}{\partial z}$	formal derivative with respect to z given by $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \jmath \frac{\partial}{\partial y}\right)$
$\frac{\partial}{\partial z^*}$	formal derivative with respect to z given by $\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \jmath \frac{\partial}{\partial y}\right)$
$\frac{\partial}{\partial \boldsymbol{Z}} f$	the gradient of f with respect to $\boldsymbol{Z} \in \mathbb{C}^{N\times Q}$ and $\frac{\partial}{\partial \boldsymbol{Z}} f \in \mathbb{C}^{N\times Q}$
$\frac{\partial}{\partial \boldsymbol{z}^T} \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{z}^*)$	formal derivatives of the vector function $\boldsymbol{f} : \mathbb{C}^{N\times 1} \times \mathbb{C}^{N\times 1} \to \mathbb{C}^{M\times 1}$ with respect to the row vector $\boldsymbol{z}^T$, and $\frac{\partial}{\partial \boldsymbol{z}^T} \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{z}^*) \in \mathbb{C}^{M\times N}$
$\frac{\partial}{\partial \boldsymbol{z}^H} \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{z}^*)$	formal derivatives of the vector function $\boldsymbol{f} : \mathbb{C}^{N\times 1} \times \mathbb{C}^{N\times 1} \to \mathbb{C}^{M\times 1}$ with respect to the row vector $\boldsymbol{z}^H$, and $\frac{\partial}{\partial \boldsymbol{z}^H} \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{z}^*) \in \mathbb{C}^{M\times N}$
π	mathematical constant, $\pi \approx 3.14159265358979323846$
a_i	i-th vector component of the vector $\boldsymbol{a}$
$a_{k,l}$	(k, l)-th element of the matrix $\boldsymbol{A}$
$\{a_0, a_1, \ldots, a_{N-1}\}$	set that contains the N elements $a_0, a_1, \ldots, a_{N-1}$
$[a_0, a_1, \ldots, a_{N-1}]$	row vector of size $1 \times N$, where the i-th elements is given by a_i
$a \cdot b,\ a \times b$	a multiplied by b
$\lVert\boldsymbol{a}\rVert$	the Euclidean norm of the vector $\boldsymbol{a} \in \mathbb{C}^{N\times 1}$, i.e., $\lVert\boldsymbol{a}\rVert = \sqrt{\boldsymbol{a}^H \boldsymbol{a}}$
$\boldsymbol{A}^{\odot k}$	the Hadamard product of $\boldsymbol{A}$ with itself k times
$\boldsymbol{A}^{-T}$	the transposed of the inverse of the invertible square matrix $\boldsymbol{A}$, i.e., $\boldsymbol{A}^{-T} = \left(\boldsymbol{A}^{-1}\right)^T$
$\boldsymbol{A}_{k,:}$	k-th row of the matrix $\boldsymbol{A}$
$\boldsymbol{A}_{:,k} = \boldsymbol{a}_k$	k-th column of the matrix $\boldsymbol{A}$

$(\boldsymbol{A})_{k,l}$	(k,l)-th component of the matrix $\boldsymbol{A}$, i.e., $(\boldsymbol{A})_{k,l} = a_{k,l}$
$\\|\boldsymbol{A}\\|_F$	the Frobenius norm of the matrix $\boldsymbol{A} \in \mathbb{C}^{N \times Q}$, i.e., $\\|\boldsymbol{A}\\|_F = \sqrt{\operatorname{Tr}\left\{\boldsymbol{A}\boldsymbol{A}^H\right\}}$
$\mathcal{A} \times \mathcal{B}$	Cartesian product of the two sets $\mathcal{A}$ and $\mathcal{B}$, that is, $\mathcal{A} \times \mathcal{B} = \{(a,b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$
arctan	inverse tangent
argmin	minimizing argument
$c_{k,l}(\boldsymbol{Z})$	the (k,l)-th cofactor of the matrix $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$
$\boldsymbol{C}(\boldsymbol{Z})$	if $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$, then the matrix $\boldsymbol{C}(\boldsymbol{Z}) \in \mathbb{C}^{N \times N}$ contains the cofactors of $\boldsymbol{Z}$
d	differential operator
$\mathcal{D}_{\boldsymbol{Z}}\boldsymbol{F}$	complex-valued matrix derivative of the matrix function $\boldsymbol{F}$ with respect to the matrix variable $\boldsymbol{Z}$
$\boldsymbol{D}_N$	duplication matrix of size $N^2 \times \frac{N(N+1)}{2}$
$\det(\cdot)$	determinant of a matrix
$\dim_{\mathbb{C}}\{\cdot\}$	complex dimension of the space it is applied to
$\dim_{\mathbb{R}}\{\cdot\}$	real dimension of the space it is applied to
$\operatorname{diag}(\cdot)$	diagonalization operator produces a diagonal matrix from a column vector
e	base of natural logarithm, $e \approx 2.71828182845904523536$
$\mathbb{E}[\cdot]$	expected value operator
$e^z = \exp(z)$	complex exponential function of the complex scalar z
$e^{\jmath \angle \boldsymbol{z}}$	if $\boldsymbol{z} \in \mathbb{C}^{N \times 1}$, then $e^{\jmath \angle \boldsymbol{z}} \triangleq \left[e^{\jmath \angle z_0}, e^{\jmath \angle z_1}, \ldots, e^{\jmath \angle z_{N-1}}\right]^T$, where $\angle z_i \in (-\pi, \pi]$ denotes the principal value of the argument of z_i
$\exp(\boldsymbol{Z})$	complex exponential matrix function, which has a complex square matrix $\boldsymbol{Z}$ as input variable
$\boldsymbol{e}_i$	standard basis in $\mathbb{C}^{N \times 1}$
$\boldsymbol{E}_{i,j}$	$\boldsymbol{E}_{i,j} \in \mathbb{C}^{N \times N}$ is given by $\boldsymbol{E}_{i,j} = \boldsymbol{e}_i \boldsymbol{e}_j^T$
$\boldsymbol{E}_{-}$	$M_t \times (m+1)N$ row-expansion of the FIR MIMO filter $\{\boldsymbol{E}(k)\}_{k=0}^{m}$, where $\boldsymbol{E}(k) \in \mathbb{C}^{M_t \times N}$
$\boldsymbol{E}_{\mid}$	$(m+1)M_t \times N$ column-expansion of the FIR MIMO filter $\{\boldsymbol{E}(k)\}_{k=0}^{m}$, where $\boldsymbol{E}(k) \in \mathbb{C}^{M_t \times N}$
$\boldsymbol{E}_{\urcorner}^{(l)}$	$(l+1)M_t \times (m+l+1)N$ matrix, which expresses the row-diagonal expanded matrix of order l of the FIR MIMO filter $\{\boldsymbol{E}(k)\}_{k=0}^{m}$, where $\boldsymbol{E}(k) \in \mathbb{C}^{M_t \times N}$
$\boldsymbol{E}_{\llcorner}^{(l)}$	$(m+l+1)M_t \times (l+1)N$ matrix, which expresses the column-diagonal expanded matrix of order l of the FIR MIMO filter $\{\boldsymbol{E}(k)\}_{k=0}^{m}$, where $\boldsymbol{E}(k) \in \mathbb{C}^{M_t \times N}$
f	complex-valued scalar function
$\boldsymbol{f}$	complex-valued vector function
$\boldsymbol{F}$	complex-valued matrix function
$\boldsymbol{F}_N$	$N \times N$ inverse DFT matrix
$f : X \to Y$	f is a function with domain X and range Y

Cambridge University Press
978-0-521-19264-4 - Complex-Valued Matrix Derivatives: With Applications in Signal Processing and Communications
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$(\cdot)^H$	$\boldsymbol{A}^H$ is the conjugate transpose of the matrix $\boldsymbol{A}$
$H(\boldsymbol{x})$	differential entropy of $\boldsymbol{x}$
$H(\boldsymbol{x} \mid \boldsymbol{y})$	conditional differential entropy of $\boldsymbol{x}$ when $\boldsymbol{y}$ is given
$I(\boldsymbol{x}; \boldsymbol{y})$	mutual information between $\boldsymbol{x}$ and $\boldsymbol{y}$
$\boldsymbol{I}$	identity matrix
$\boldsymbol{I}_p$	$p \times p$ identity matrix
$\boldsymbol{I}_N^{(k)}$	$N \times N$ matrix containing zeros everywhere and ones on the k-th diagonal where the lower diagonal is numbered as $N - 1$, the main diagonal is numbered with 0, and the upper diagonal is numbered with $-(N - 1)$
$\mathrm{Im}\{\cdot\}$	returns imaginary part of the input
$\jmath$	imaginary unit
$\boldsymbol{J}$	$MN \times MN$ matrix with $N \times N$ identity matrices on the main reverse block diagonal and zeros elsewhere, i.e., $\boldsymbol{J} = \boldsymbol{J}_M \otimes \boldsymbol{I}_N$
$\boldsymbol{J}_N$	$N \times N$ reverse identity matrix with zeros everywhere except $+1$ on the main reverse diagonal
$\boldsymbol{J}_N^{(k)}$	$N \times N$ matrix containing zeros everywhere and ones on the k-th reverse diagonal where the upper reverse is numbered by $N - 1$, the main reverse diagonal is numbered with 0, and the lower reverse diagonal is numbered with $-(N - 1)$
$\mathbb{K}^{N \times Q}$	$N \times Q$ dimensional vector space over the field $\mathbb{K}$ and possible values of $\mathbb{K}$ are, for example, $\mathbb{R}$ or $\mathbb{C}$
$\boldsymbol{K}_{Q,N}$	commutation matrix of size $QN \times QN$
$\mathcal{L}$	Lagrange function
$\boldsymbol{L}_d$	$N^2 \times N$ matrix used to place the diagonal elements of $\boldsymbol{A} \in \mathbb{C}^{N \times N}$ on $\mathrm{vec}(\boldsymbol{A})$
$\boldsymbol{L}_l$	$N^2 \times \frac{N(N-1)}{2}$ matrix used to place the elements strictly below the main diagonal of $\boldsymbol{A} \in \mathbb{C}^{N \times N}$ on $\mathrm{vec}(\boldsymbol{A})$
$\boldsymbol{L}_u$	$N^2 \times \frac{N(N-1)}{2}$ matrix used to place the elements strictly above the main diagonal of $\boldsymbol{A} \in \mathbb{C}^{N \times N}$ on $\mathrm{vec}(\boldsymbol{A})$
$\lim_{z \to a} f(z)$	limit of $f(z)$ when z approaches a
$\ln(z)$	principal value of natural logarithm of z, where $z \in \mathbb{C}$
$m_{k,l}(\boldsymbol{Z})$	the (k, l)-th minor of the matrix $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$
$\boldsymbol{M}(\boldsymbol{Z})$	if $\boldsymbol{Z} \in \mathbb{C}^{N \times N}$, then the matrix $\boldsymbol{M}(\boldsymbol{Z}) \in \mathbb{C}^{N \times N}$ contains the minors of $\boldsymbol{Z}$
max	maximum value of
min	minimum value of
$\mathbb{N}$	natural numbers $\{1, 2, 3, \ldots\}$
$n!$	factorial of n given by $n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$
$\mathrm{perm}(\cdot)$	permanent of a matrix
$\boldsymbol{P}_N$	primary circular matrix of size $N \times N$
$\mathbb{R}$	the set of real numbers

$\mathbb{R}^+$	the set $(0, \infty)$
$\text{rank}(\cdot)$	rank of a matrix
$\text{Re}\{\cdot\}$	returns real part of the input
$(\cdot)^T$	$\boldsymbol{A}^T$ is the transpose of the matrix $\boldsymbol{A}$
$\mathcal{T}^{(k)}\{\cdot\}$	linear reshaping operator used in connection with transmitter FIR MIMO optimization
$\text{Tr}\{\cdot\}$	trace of a square matrix
$v(\cdot)$	return all the elements on and below main diagonal taken in the same column-wise order as the ordinary vec-operator
$\text{vec}(\cdot)$	vectorization operator stacks the columns into a long column vector
$\text{vec}_d(\cdot)$	extracts the diagonal elements of a square matrix and returns them in a column vector
$\text{vec}_l(\cdot)$	extracts the elements strictly below the main diagonal of a square matrix in a column-wise manner and returns them into a column vector
$\text{vec}_u(\cdot)$	extracts the elements strictly above the main diagonal of a square matrix in a row-wise manner and returns them into a column vector
$\text{vecb}(\cdot)$	block vectorization operator stacks square block matrices of the input into a long block column matrix
$\boldsymbol{V}$	permutation matrix of size $\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}$ given by $\boldsymbol{V} = [\boldsymbol{V}_d, \ \boldsymbol{V}_l]$
$\boldsymbol{V}_d$	matrix of size $\frac{N(N+1)}{2} \times N$ used to place the elements of $\text{vec}_d(\boldsymbol{A})$ on $v(\boldsymbol{A})$, where $\boldsymbol{A} \in \mathbb{C}^{N \times N}$ is symmetric
$\boldsymbol{V}_l$	matrix of size $\frac{N(N+1)}{2} \times \frac{N(N-1)}{2}$ used to place the elements of $\text{vec}_l(\boldsymbol{A})$ on $v(\boldsymbol{A})$, where $\boldsymbol{A} \in \mathbb{C}^{N \times N}$ is symmetric
$\mathcal{W}$	set containing matrices in a manifold
$\mathcal{W}^*$	set containing all the complex conjugate elements of the elements in $\mathcal{W}$, that is, when $\mathcal{W}$ is given, $\mathcal{W}^* \triangleq \{\boldsymbol{W}^* \mid \boldsymbol{W} \in \mathcal{W}\}$
$\boldsymbol{W}$	symbol often used to represent a matrix in a manifold, that is, $\boldsymbol{W} \in \mathcal{W}$, where $\mathcal{W}$ represents a manifold
$\tilde{\boldsymbol{W}}$	matrix used to represent a matrix of the same size as the matrix $\boldsymbol{W}$; however, the matrix $\tilde{\boldsymbol{W}}$ is unpatterned
$[x_0, x_1]$	closed interval given by the set $\{x \mid x_0 \leq x \leq x_1\}$
$(x_0, x_1]$	semi-open interval given by the set $\{x \mid x_0 < x \leq x_1\}$
(x_0, x_1)	open interval given by the set $\{x \mid x_0 < x < x_1\}$
$\boldsymbol{x}(n)_{\text{!`}}^{(\nu)}$	column-expansion of vector time-series of size $(\nu + 1)N \times 1$, where $\boldsymbol{x}(n) \in \mathbb{C}^{N \times 1}$
$\mathbb{Z}$	the set of integers, that is, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}_N$	the set $\{0, 1, \ldots, N-1\}$
z	complex-valued scalar variable
$\boldsymbol{z}$	complex-valued vector variable
$\boldsymbol{Z}$	complex-valued matrix variable