1 Formulation of the Equations of Motion

The first step in the analysis of any structural vibration problem is the formulation of the equations of motion. It is an important part of the exercise, since the success of the analysis is dependent upon the equations of motion being formulated correctly. This process will be less prone to errors if a routine procedure for formulating the equations can be established. In this chapter a number of methods will be presented and discussed.

1.1 Dynamic Equilibrium

The equations of motion of any dynamic system can be written down using Newton's second law of motion, which states that 'the rate of change of momentum of a mass is equal to the force acting on it'.

Consider a mass, m, which is displaced a distance $u(t)$ when acted upon by a force $f(t)$, both being functions of time, t, as shown in Figure 1.1, then Newton's second law of motion gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right) = f(t) \tag{1.1}$$

For constant m, which will be assumed throughout this book, equation (1.1) reduces to

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = f \tag{1.2}$$

or

$$m\ddot{u} = f \tag{1.3}$$

where dots denote differentiation with respect to time.

Equation (1.3) can be rewritten in the form

$$f - m\ddot{u} = 0 \tag{1.4}$$

If the term $-m\ddot{u}$ is now regarded as a force, then equation (1.4) represents an equation of equilibrium, that is, the sum of the forces acting on the mass is equal to zero. The introduction of this fictitious force, which is referred to as an inertia force, of magnitude $m\ddot{u}$, acting in the opposite direction to the acceleration, $\ddot{u}$, allows an
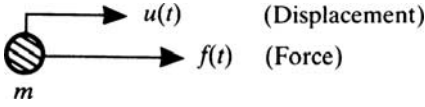
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Figure 1.1. Motion of a single mass.

equation of dynamic equilibrium to be formulated using the concepts of static equi-
librium. This equation of dynamic equilibrium, when rearranged, gives the equation
of motion of the system. This concept is known as d'Alembert's principle.

EXAMPLE 1.1 Derive the equation of motion of the single mass, spring, damper
system shown in Figure 1.2(a).

The forces acting on the mass consist of the externally applied force f, a
restoring force ku due to the spring, a damping force $c\dot{u}$ due to the viscous
damper and a fictitious inertia force $m\ddot{u}$. All act in the directions shown in
Figure 1.2(b). For equilibrium

$$-m\ddot{u} - c\dot{u} - ku + f = 0 \tag{1.5}$$

Rearranging, gives the equation of motion

$$m\ddot{u} + c\dot{u} + ku = f \tag{1.6}$$

The above concepts can be extended to multi-degree of freedom systems. Consider a
system of N masses. The equations of dynamic equilibrium are obtained by equating
the sums of the forces and moments on each mass of the system to zero. This gives

$$\vec{f}_j - \frac{\mathrm{d}}{\mathrm{d}t}(m_j \dot{\vec{u}}_j) = 0 \qquad j = 1, 2, \ldots, N \tag{1.7}$$

and

$$\vec{L}_j - \frac{\mathrm{d}}{\mathrm{d}t}(\vec{J}_j) = 0 \qquad j = 1, 2, \ldots, N \tag{1.8}$$

In these equations $\vec{u}_j$ is the displacement of the mass m_j, $\vec{f}_j$ is the sum of the applied
forces, $\vec{J}_j$ is the angular momentum, and $\vec{L}_j$ is the sum of the applied moments. If
the vectors $\vec{u}_j$ do not represent independent motions, equations (1.7) and (1.8) must
be modified by constraints of the form

$$g_j(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N) = 0 \qquad j = 1, 2, \ldots, m \tag{1.9}$$

where m is the number of constraints. This aspect is discussed in Section 1.5.

EXAMPLE 1.2 Derive the equations of motion of the system shown in Figure 1.3.

The mass m_1 has two forces acting on it due to the extension of the two
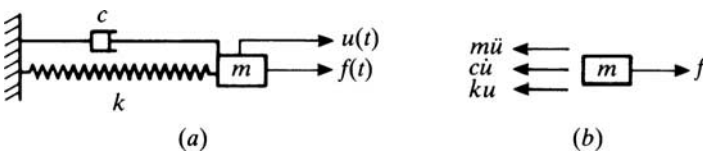springs joining it to the masses m_2 and m_3.

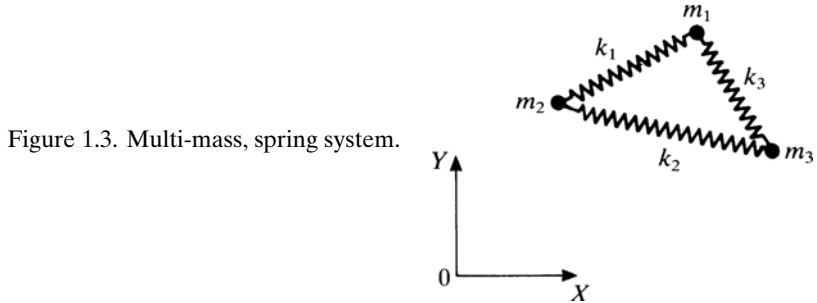Figure 1.2. Single mass, spring, damper system.

Figure 1.3. Multi-mass, spring system.

If the position vectors of m_1 and m_2 are $\vec{V}1$ and $\vec{V}2$ respectively, then the unit vector $\vec{n}_1$, along the line 2–1 is

$$\vec{n}_1 = \frac{1}{L_1}(\vec{V}1 - \vec{V}2) \tag{1.10}$$

where

$$L_1 = \text{abs}(\vec{V}1 - \vec{V}2)$$

If the displacements of m_1 and m_2 are denoted by $\vec{U}_1$ and $\vec{U}_2$ then the extension, e_1, of the spring joining m_1 and m_2 is given by the scalar product

$$e_1 = (\vec{U}_1 - \vec{U}_2) \cdot \vec{n}_1 \tag{1.11}$$

If the stiffness of the spring is k_1, then the force, f_1, acting on the mass m_1 in the direction $\vec{n}_1$ is

$$f_1 = -k_1 e_1 = k_1(\vec{U}_2 - \vec{U}_1) \cdot \vec{n}_1 \tag{1.12}$$

Similarly, the force, f_3, acting on the mass m_1 in the direction $\vec{n}_3$ is

$$f_3 = k_3(\vec{U}_3 - \vec{U}_1) \cdot \vec{n}_3 \tag{1.13}$$

where

$$\vec{n}_3 = \frac{1}{L_3}(\vec{V}1 - \vec{V}3) \tag{1.14}$$

and

$$L_3 = \text{abs}(\vec{V}1 - \vec{V}3).$$

The equation of dynamic equilibrium for m_1 is therefore

$$f_1\vec{n}_1 + f_3\vec{n}_3 - m_1\ddot{\vec{U}}_1 = 0 \tag{1.15}$$

When the components of each of the vectors are substituted in this equation, two scalar equations will be obtained. These can then be rearranged, in the manner shown in Example 1.1, to give the equations of motion of the mass m_1. The equations of motion of the masses m_2 and m_3 are obtained in a similar way.

1.2 Principle of Virtual Displacements

If the structure to be analysed is a complex one, then the vectoral addition of all the forces acting at each mass point is difficult. This difficulty may be overcome by first

using d'Alembert's principle and then the principle of virtual displacements. By this means the equations of dynamic equilibrium and hence the equations of motion, are formulated indirectly.

The principle of virtual displacements states that 'if a system, which is in equilibrium under the action of a set of forces, is subjected to a virtual displacement, then the total work done by the forces will be zero'. In this context, a virtual displacement is a physically possible one, that is, any displacement which is compatible with the system constraints.

EXAMPLE 1.3 Use the principle of virtual displacements to derive the equation of motion of the system shown in Figure 1.2.

Figure 1.2(*b*) shows the forces acting after the application of d'Alembert's principle. If the system is given a virtual displacement δu, then the principle of virtual displacements gives

$$-m\ddot{u}\delta u - c\dot{u}\delta u - ku\delta u + f\delta u = 0 \tag{1.16}$$

Rearranging gives

$$\left(-m\ddot{u} - c\dot{u} - ku + f\right)\delta u = 0 \tag{1.17}$$

Since δu is arbitrary and non-zero, then

$$m\ddot{u} + c\dot{u} + ku = f \tag{1.18}$$

The advantage of this approach is that the virtual work contributions are scalar quantities which can be added algebraically.

For a multi-degree of freedom system, the principle of virtual work gives

$$\sum_{j=1}^{N}\left(\vec{f}_j - \frac{\mathrm{d}}{\mathrm{d}t}(m_j\dot{\vec{u}}_j)\right)\cdot\delta\vec{u}_j + \sum_{j=1}^{N}\left(\vec{L}_j - \frac{\mathrm{d}}{\mathrm{d}t}(\vec{J}_j)\right)\cdot\delta\vec{\theta}_j = 0 \tag{1.19}$$

where the $\delta\vec{u}_j$ are virtual displacements and the $\delta\vec{\theta}_j$ virtual rotations. Since each of these is arbitrary, equations (1.7) and (1.8) must hold.

1.3 Hamilton's Principle

Although the principle of virtual displacements overcomes the problem of vectorial addition of forces, virtual work itself is calculated from the scalar product of two vectors, one representing a force and one a virtual displacement. This disadvantage can be largely overcome by using Hamilton's principle to determine the equations of motion.

Consider a mass, m, which is acted upon by a force, f_T, causing a displacement, u, as shown in Figure 1.4. f_T represents the sum of all the applied forces, both conservative and non-conservative.

The work done by a conservative force in moving a mass from one point to another depends only on the position of the two points and is independent of the
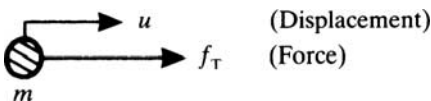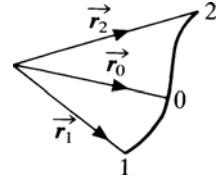

Figure 1.4. Motion of a single mass.

Figure 1.5. Path taken by a mass.

path taken between them. The work done by non-conservative forces does depend upon the path taken between the two points. Non-conservative forces are energy dissipating forces such as friction forces, or forces imparting energy to the system such as external forces.

The work done by a conservative force can be obtained from the change in potential energy. The potential energy $V(\vec{r})$ associated with position $\vec{r}$ is defined as the work done by a conservative force $\vec{f}$ in moving a mass from position $\vec{r}$ to a reference position $\vec{r}_0$. That is

$$V(\vec{r}) = \int_{\vec{r}}^{\vec{r}_0} \vec{f} \cdot \mathrm{d}\vec{r} \tag{1.20}$$

The work done by a conservative force $\vec{f}$ in moving a mass from position $\vec{r}_1$ to position $\vec{r}_2$, as shown in Figure 1.5, is

$$\begin{aligned} W &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{f} \cdot \mathrm{d}\vec{r} \\ &= \int_{\vec{r}_1}^{\vec{r}_0} \vec{f} \cdot \mathrm{d}\vec{r} - \int_{\vec{r}_2}^{\vec{r}_0} \vec{f} \cdot \mathrm{d}\vec{r} \\ &= -\{V(\vec{r}_2) - V(\vec{r}_1)\} \end{aligned} \tag{1.21}$$

Since the force is a conservative one, the work done is independent of the path, and so in Figure 1.5 the path has been chosen to pass through the reference point 0.

Equation (1.21) states that the work done by a conservative force is minus the change in potential energy. In differential form this is

$$\delta W = -\delta V \tag{1.22}$$

The type of potential energy which will be considered in this book is the elastic potential energy, or strain energy U.

Consider a linear elastic spring of stiffness, k, which is stretched by an amount u. Then the force, f, in the spring in the direction of u is

$$f = -ku \tag{1.23}$$

and the potential energy

$$U = \int_u^0 f \, \mathrm{d}u = -\int_u^0 ku \, \mathrm{d}u = \frac{1}{2}ku^2 \tag{1.24}$$

Applying the principle of virtual displacements to the system in Figure 1.4 gives
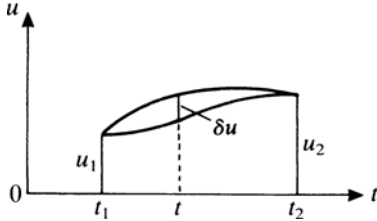
$$f_{\mathrm{T}}\delta u - m\ddot{u}\delta u = 0 \tag{1.25}$$

Figure 1.6. Variation in the motion of a mass.

where δu is a virtual displacement.

$$\text{Now } f_{\mathrm{T}}\delta u = \delta W = \text{work done by the forces} \tag{1.26}$$

and

$$m\ddot{u}\delta u = m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}\delta u) - m\dot{u}\delta\dot{u} \tag{1.27}$$

where it has been assumed that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta u) = \delta\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) = \delta\dot{u}$$

Equation (1.27) can be further modified as follows

$$\begin{aligned}
m\ddot{u}\delta u &= m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}\delta u) - \delta\left(\frac{1}{2}m\dot{u}^2\right) \\
&= m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}\delta u) - \delta T
\end{aligned} \tag{1.28}$$

where

$$T = \tfrac{1}{2}m\dot{u}^2 \tag{1.29}$$

represents the kinetic energy of the system.

Substituting equations (1.26) and (1.28) into equation (1.25) gives

$$\delta W - m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}\delta u) + \delta T = 0$$

or, on rearranging

$$\delta T + \delta W = m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}\delta u) \tag{1.30}$$

If the position of the mass is known at two instants of time t_1 and t_2, then its motion during this interval of time can be represented by a curve, as shown in Figure 1.6. A slightly different curve or path is obtained if, at any instant, a small variation in position δu is allowed with no associated change in time; that is $\delta t = 0$ (Figure 1.6). The stipulation is made, however, that at times t_1 and t_2 the two paths coincide, that is

$$\delta u = 0 \qquad \text{at } t = t_1 \text{ and } t = t_2 \tag{1.31}$$

The problem is to choose the true path from u_1 to u_2 from all the possible ones.

Multiplying equation (1.30) by dt and integrating between t_1 and t_2 gives

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} m \frac{d}{dt}(\dot{u}\delta u)\, dt$$
$$= [m\dot{u}\delta u]_{t_1}^{t_2} = 0 \tag{1.32}$$

by virtue of equation (1.31). Equation (1.32), therefore, states that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0 \tag{1.33}$$

Separating the forces into conservative and non-conservative forces, gives

$$\delta W = \delta W_c + \delta W_{nc} \tag{1.34}$$

Using equation (1.22), namely,

$$\delta W_c = -\delta V \tag{1.35}$$

equation (1.34) becomes

$$\delta W = -\delta V + \delta W_{nc} \tag{1.36}$$

Substituting equation (1.36) into equation (1.33) gives

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\, dt = 0 \tag{1.37}$$

or

$$\int_{t_1}^{t_2} (\delta(T - V) + \delta W_{nc})\, dt = 0 \tag{1.38}$$

Note that equation (1.37) cannot be written in the form

$$\int_{t_1}^{t_2} \delta(T - V + W_{nc})\, dt = 0 \tag{1.39}$$

since a work function W_{nc} does not exist for non-conservative forces. However, the virtual work can always be calculated. Equation (1.38) is the mathematical statement of Hamilton's principle. For a conservative system $\delta W_{nc} = 0$. In this case equation (1.38) shows that the integral of $(T - V)$ along the true time path is stationary. It can be shown, for the applications considered in this book, that the stationary value of the integral is a minimum.

The application of this principle leads directly to the equations of motion for any system. It can be applied to both discrete, multi-degree of freedom systems (as shown in Appendix 1) and continuous systems (as illustrated in Section 2.11). The advantage of this formulation is that it uses scalar energy quantities. Vector quantities may only be required in calculating the work done by the non-conservative forces. As previously stated, the only potential energy of interest in this book is elastic strain energy U. The form of Hamilton's principle to be used is therefore

$$\int_{t_1}^{t_2} (\delta(T - U) + \delta W_{nc})\, dt = 0 \tag{1.40}$$

EXAMPLE 1.4 Use Hamilton's principle to derive the equations of motion of the system shown in Figure 1.2.

For this system

$$T = \tfrac{1}{2}m\dot{u}^2$$
$$U = \tfrac{1}{2}ku^2 \tag{1.41}$$
$$\delta W_{\text{nc}} = f\delta u - c\dot{u}\delta u$$

Substituting into equation (1.40) gives

$$\int_{t_1}^{t_2} \delta\left(\frac{1}{2}m\dot{u}^2 - \frac{1}{2}ku^2\right)\,\mathrm{d}t + \int_{t_1}^{t_2}(f\delta u - c\dot{u}\delta u)\,\mathrm{d}t = 0 \tag{1.42}$$

that is

$$\int_{t_1}^{t_2}(m\dot{u}\delta\dot{u} - ku\delta u + f\delta u - c\dot{u}\delta u)\,\mathrm{d}t = 0 \tag{1.43}$$

Now

$$\delta\dot{u} = \delta\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\delta u)$$

Hence integrating the first term by parts gives

$$\int_{t_1}^{t_2} m\dot{u}\delta\dot{u}\,\mathrm{d}t = [m\dot{u}\delta u]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{u}\delta u\,\mathrm{d}t$$
$$= -\int_{t_1}^{t_2} m\ddot{u}\delta u\,\mathrm{d}t \tag{1.44}$$

by virtue of equation (1.31).

Substituting equation (1.44) into equation (1.43) gives

$$\int_{t_1}^{t_2}(-m\ddot{u} - c\dot{u} - ku + f)\delta u\,\mathrm{d}t = 0 \tag{1.45}$$

Since δu is arbitrary, equation (1.45) is satisfied only if

$$m\ddot{u} + c\dot{u} + ku = f \tag{1.46}$$

1.4 Lagrange's Equations

When Hamilton's principle is applied to discrete systems it can be expressed in a more convenient form. To illustrate this, consider the system shown in Figure 1.2. The kinetic and strain energies are given by

$$T = \tfrac{1}{2}m\dot{u}^2 = T(\dot{u}), \qquad U = \tfrac{1}{2}ku^2 = U(u) \tag{1.47}$$

and the virtual work done by the non-conservative forces is

$$\delta W_{\text{nc}} = (f - c\dot{u})\delta u \tag{1.48}$$

Equation (1.40) therefore becomes

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial \dot{u}}\delta\dot{u} - \frac{\partial U}{\partial u}\delta u + (f - c\dot{u})\delta u\right)\mathrm{d}t = 0 \tag{1.49}$$

Integrating the first term by parts gives

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{u}} \delta \dot{u} \, \mathrm{d}t = \left[\frac{\partial T}{\partial \dot{u}} \delta u \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{u}} \right) \delta u \, \mathrm{d}t$$

$$= - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{u}} \right) \delta u \, \mathrm{d}t \tag{1.50}$$

as a consequence of using equation (1.31).

Substituting equation (1.50) into equation (1.49) gives

$$\int_{t_1}^{t_2} \left\{ -\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{u}} \right) - \frac{\partial U}{\partial u} + f - c\dot{u} \right\} \delta u \, \mathrm{d}t = 0 \tag{1.51}$$

Since δu is arbitrary, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{u}} \right) + \frac{\partial U}{\partial u} + c\dot{u} = f \tag{1.52}$$

Introducing a dissipation function, D, which is defined by

$$D = \tfrac{1}{2} c\dot{u}^2 \tag{1.53}$$

the damping force is given by

$$c\dot{u} = \frac{\partial D}{\partial \dot{u}} \tag{1.54}$$

The dissipation function represents the instantaneous rate of energy dissipation which is given by

$$\tfrac{1}{2} \times \text{damping force} \times \text{rate of extension of damper}$$

Substituting the relationship (1.54) into equation (1.52) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{u}} \right) + \frac{\partial D}{\partial \dot{u}} + \frac{\partial U}{\partial u} = f \tag{1.55}$$

Equation (1.55) is Lagrange's equation for a single degree of freedom system. Substituting equations (1.47) and (1.53) into equation (1.55) gives

$$m\ddot{u} + c\dot{u} + ku = f \tag{1.56}$$

which is the equation of motion of the system. It can be seen that the term $(\mathrm{d}/\mathrm{d}t)(\partial T/\partial \dot{u})$ gives the inertia force and $\partial U/\partial u$ the restoring force due to the spring.

In the case of a multi-degree of freedom system, the deformation of which is described by n independent displacements $q_1, q_2, \ldots, q_n$, then the kinetic energy is a function of the velocities $\dot{q}_j (j = 1, 2, \ldots, n)$ only and the strain energy a function of the displacements $q_j (j = 1, 2, \ldots, n)$ only, that is

$$T = T(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

$$U = U(q_1, q_2, \ldots, q_n) \tag{1.57}$$

Similarly, the dissipation function is a function of the velocities $\dot{q}_j$, that is

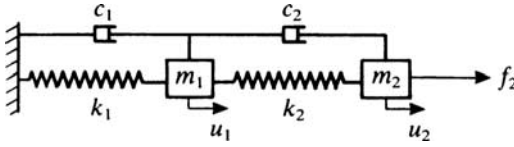$$D = D(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \tag{1.58}$$

Figure 1.7. Two degree of freedom mass, spring, damper system.

Also, the work done by the non-conservative forces can be written in the form (see Appendix 1)

$$\delta W_{nc} = \sum_{j=1}^{n} \left(Q_j - \frac{\partial D}{\partial \dot{q}_j} \right) \delta q_j \tag{1.59}$$

where the Q_j are generalised forces.

Lagrange's equations now take the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial D}{\partial \dot{q}_j} + \frac{\partial U}{\partial q_j} = Q_j, \qquad j = 1, 2, \ldots, n \tag{1.60}$$

These equations are derived in Appendix 1.

EXAMPLE 1.5 Use Lagrange's equations to derive the equations of motion of the system shown in Figure 1.7

The kinetic energy is given by

$$T = \tfrac{1}{2} m_1 \dot{u}_1^2 + \tfrac{1}{2} m_2 \dot{u}_2^2 \tag{1.61}$$

the dissipation function by

$$\begin{aligned} D &= \tfrac{1}{2} c_1 \dot{u}_1^2 + \tfrac{1}{2} c_2 (\dot{u}_2 - \dot{u}_1)^2 \\ &= \tfrac{1}{2} (c_1 + c_2) \dot{u}_1^2 - c_2 \dot{u}_1 \dot{u}_2 + \tfrac{1}{2} c_2 \dot{u}_2^2 \end{aligned} \tag{1.62}$$

and the strain energy by

$$\begin{aligned} U &= \tfrac{1}{2} k_1 u_1^2 + \tfrac{1}{2} k_2 (u_2 - u_1)^2 \\ &= \tfrac{1}{2} (k_1 + k_2) u_1^2 - k_2 u_1 u_2 + \tfrac{1}{2} k_2 u_2^2 \end{aligned} \tag{1.63}$$

The virtual work done by the applied force is

$$\delta W = f_2 \delta u_2 \tag{1.64}$$

Applying Lagrange's equations (1.60) gives

$$\begin{aligned} m_1 \ddot{u}_1 + (c_1 + c_2) \dot{u}_1 - c_2 \dot{u}_2 + (k_1 + k_2) u_1 - k_2 u_2 = 0 \\ m_2 \ddot{u}_2 - c_2 \dot{u}_1 + c_2 \dot{u}_2 - k_2 u_1 + k_2 u_2 = f_2 \end{aligned} \tag{1.65}$$

The procedure can be made even more systematic, and therefore less prone to errors, by using matrix notation. The kinetic energy, dissipation function and strain energy can all be written in the following forms

$$\begin{aligned} T &= \tfrac{1}{2} \{\dot{\mathbf{q}}\}^{\mathrm{T}} [\mathbf{M}] \{\dot{\mathbf{q}}\} \\ D &= \tfrac{1}{2} \{\dot{\mathbf{q}}\}^{\mathrm{T}} [\mathbf{C}] \{\dot{\mathbf{q}}\} \\ U &= \tfrac{1}{2} \{\mathbf{q}\}^{\mathrm{T}} [\mathbf{K}] \{\mathbf{q}\} \end{aligned} \tag{1.66}$$