1
Introduction

Interest in magnetoconvection arose initially from astrophysics, following the discovery of strong magnetic fields in sunspots, and the realization that their relative coolness (and hence their darkness) was a consequence of magnetic interference with convection. As theoretical studies progressed from linear to nonlinear investigations, and ultimately to massive numerical experiments, it became clear not only that magnetoconvection poses in itself a fascinating challenge to applied mathematicians but also that it serves as a prototype of double-diffusive behaviour in fluid dynamics, oceanography and laboratory experiments.

In this opening chapter we first summarize the development of our subject and then provide a brief survey of the chapters that follow in the book. Although we shall focus our attention on idealized configurations that are mathematically tractable, we also discuss more complex behaviour in the real world.

1.1 Background and motivation

The original motivation for our subject came from astrophysics. Stars like the Sun, with deep outer convection zones, are magnetically active. Their magnetic fields are maintained by hydromagnetic dynamo action, resulting from interactions between convection, rotation and magnetic fields in their interiors – just as the geomagnetic field is maintained by a dynamo in the Earth's liquid core. The most prominent magnetic features on the Sun are sunspots, like that shown in Figure 1.1. Although such a spot covers less than 1% of the solar disc, there are other more active stars with huge spots that spread over significant fractions of their surfaces (Thomas and Weiss 2008). Modern astrophysics began with the development of spectroscopy: exploiting the Zeeman splitting of spectral lines by a magnetic field, Hale (1908) discovered that sunspots are in fact the sites of strong magnetic fields (up

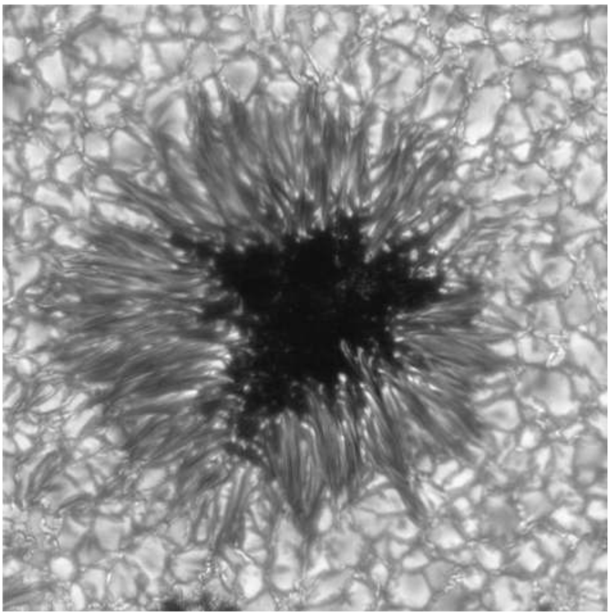

Figure 1.1 High-resolution G-band image of a symmetrical sunspot, obtained with the Swedish Solar Telescope on La Palma. The magnetic field is vertical at the centre of the spot but becomes increasingly inclined towards the periphery. In the central dark umbra there is a tesseral pattern of convection, with isolated bright dots, a few of which are visible. The penumbra has a filamentary structure, with roll-like patterns of convection. The small-scale cellular pattern surrounding the spot is the photospheric granulation. Hot plasma rises in the centre of a granule and cooler fluid sinks around its periphery. The bright points nestling between granules indicate the presence of small-scale magnetic fields. (Courtesy of L. Rouppe van der Voort and the Royal Swedish Academy of Sciences.)

to 0.3 tesla, or 3000 gauss). By the 1930s it had been realized that radiative energy transport in the interior of the Sun, and of similar stars, gives way to convective transport near their surfaces. The solar convection zone manifests itself as small-scale cellular convection ('granulation') at the photosphere, as can be seen in Figure 1.1 and, in greater detail, in Figure 1.2. Intense small-scale magnetic fields are concentrated in the network of cool sinking fluid that encloses the bright granules.

The prehistory of magnetoconvection began with an exchange of letters between two astrophysicists, Ludwig Biermann in Germany and Thomas Cowling in England, in 1938–39. Biermann suggested that the coolness of a spot was caused by magnetic inhibition of convection (Cowling 1985). Cowling (characteristically) expressed initial doubts but, after comparing

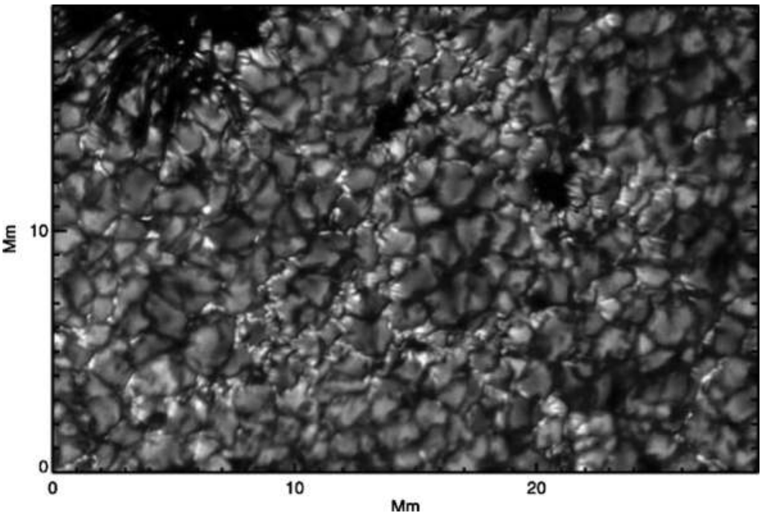

Figure 1.2 Small-scale magnetic fields in an active region on the Sun, shown up as bright points in a G-band image. The dark pores contain stronger fields, associated with the sunspot at top left. Magnetic fields are swept aside by rising and expanding plasma in the centres of the granules, and concentrated at their edges. The individual cells have diameters of around 1500 km. (From Berger, Rouppe van der Voort, and Löfdahl 2007 © AAS. Reproduced with permission.)

Reynolds and Maxwell stresses, conceded that Biermann's idea was correct. The war interrupted their correspondence but Biermann (1941) published a brief statement of his idea, arguing that the magnetic field in a sunspot was strong enough to suppress convection, since the magnetic energy density was locally an order of magnitude greater than the kinetic energy density of granular convection.[1] After the war, it was Cowling (1953) who drew attention to this obscure reference and helped to develop Biermann's ideas into a more coherent theory.

Meanwhile, Alfvén (1942a) had ushered in the new subject of magnetohydrodynamics (MHD) by reporting the existence of transverse waves (now called after him) in a highly conducting fluid, and then going on to describe the magnetic field as 'frozen in' to a perfectly conducting liquid (Alfvén 1942b; Ferraro and Plumpton 1961).[2] This concept was further

[1] Biermann ascribed this criterion to Cowling – but cautiously referred to a paper that does not mention it, rather than to their private correspondence. The key paragraph of Biermann's 1941 paper, copies of which are hard to find, has been reproduced (with a translation) by Thomas and Weiss (1992).

[2] Alfvén relied on a physical argument: 'Every motion (perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it can give infinite eddy currents. Thus the matter of the liquid is fastened to the lines of force'. What is now known as Alfvén's Theorem does not appear in the first edition of his book (Alfvén 1950).

developed by his student Walén (1946), who showed that $\boldsymbol{B}/\rho$, where $\boldsymbol{B}$ is the magnetic field and ρ is the density, evolves in the same way as a line element moving with the fluid. Within the next few years it was realized that magnetic fields in a perfectly conducting fluid behave analogously to vorticity in an inviscid fluid, so that, corresponding to Kelvin's Theorem in fluid dynamics, there is Alfvén's Theorem: the magnetic flux through a surface moving with the fluid is conserved (e.g. Lundquist 1952). Shortly afterwards, the existence of MHD waves was demonstrated experimentally, first in mercury and then in liquid sodium (Lehnert 1954). Meanwhile, interest in MHD had been stimulated by the first attempts to describe the generation of the geomagnetic field by hydromagnetic dynamo action in the Earth's molten core.

A preliminary attempt to quantify the stabilizing effect of a magnetic field on convection was made by Walén (1949).[3] In modern terminology, Walén considered an unstably stratified layer with a superadiabatic temperature gradient β and a horizontal field B_0. The upward buoyancy force on a fluid element displaced a distance ξ from its equilibrium position is then $g\rho\alpha\beta\xi$, where α is the coefficient of thermal expansion; this is opposed by the curvature force $B_0^2\xi/\mu_0 l^2$, where l is the semi-wavelength of the disturbance and μ_0 is the permeability of free space. Thus convection is suppressed if $g\alpha\beta < B_0^2/(\mu_o \rho l^2)$ (Cowling 1953).

The 1950s saw the development of linear stability analysis, culminating with the publication of the first edition of Cowling's concise book in 1957 and of Chandrasekhar's tome in 1961. Cowling (1957, 1976a) rendered Walén's argument more precise. He considered two-dimensional disturbances to a vertical field $\boldsymbol{B}_0$ in a perfectly conducting fluid. For rolls of width l and depth d there is a transition (the 'exchange of stabilities') from undamped oscillations to overturning convection as β is increased. Then convection sets in for

$$g\alpha\beta > \pi^2(l^2 + d^2)\frac{B_0^2}{\mu_0 \rho d^4}. \qquad (1.1)$$

A more realistic – and more interesting – situation arises when non-ideal diffusive effects are included. Thompson (1951) introduced a magnetic diffusivity η and a thermal diffusivity κ: linear behaviour then depends critically on their ratio $\zeta = \eta/\kappa$.[4] In a star, where radiative diffusion predominates, ζ is typically very small. For $\zeta > 1$ and $|B_0|$ very large the static conducting solution becomes unstable to monotonically growing modes for

[3] Walén's publication was mainly concerned with interactions between magnetic fields and rotation. It was published privately, as Walén had fallen out with his superiors.
[4] This ratio is the reciprocal of the Roberts number, used in geophysics.

$$g\alpha\beta > \frac{\pi^2(l^2+d^2)}{\zeta}\frac{B_0^2}{\mu_0\rho d^4}\,. \tag{1.2}$$

Thus diffusion facilitates overturning convection by allowing field lines to slip through the fluid. With $\zeta < 1$ and $|B_0|$ very large, however, convection occurs as growing (overstable) oscillations for

$$g\alpha\beta > \pi^2\zeta(l^2+d^2)\frac{B_0^2}{\mu_0\rho d^4}\,; \tag{1.3}$$

for $\zeta \ll 1$ this happens at a much lower value of β than that given by Equation (1.1).

Chandrasekhar (1952, 1961) added a viscous diffusivity ν and provided an exhaustive treatment of linear stability theory in terms of the Rayleigh number R and the parameter Q, the square of a Hartmann number, now known as the Chandrasekhar number, where

$$R = \frac{g\alpha\beta d^4}{\kappa\nu} \qquad \text{and} \qquad Q = \frac{B_0^2 d^2}{\mu_0\rho\eta\nu}. \tag{1.4}$$

In particular, he established the critical Rayleigh numbers for which convection can set in as overturning modes (at a stationary bifurcation) or as oscillatory modes (at an oscillatory or Hopf bifurcation). Experiments involving a layer of mercury yielded measurements of the critical Rayleigh number for the onset of convection, Ra_c, as a function of Q, that were consistent with these linear predictions (Nakagawa 1955, 1957).

The next issue is what happens after the initial onset of convection, whether this occurs at a stationary or an oscillatory bifurcation. This book is mainly concerned with investigations of nonlinear behaviour, using a combination of analytical and numerical techniques. The real theoretical breakthrough came in the 1980s,[5] with the development of nonlinear dynamics and bifurcation theory. That made it possible to understand the patterns of behaviour that were gradually being revealed by ever more complicated numerical experiments. These began with studies of kinematic flux expulsion; then, as computers grew more powerful, it became feasible to model two-dimensional magnetoconvection in an incompressible (Boussinesq) fluid.

The key theoretical development was the analysis of behaviour near a degenerate Takens–Bogdanov bifurcation, where oscillatory and stationary bifurcations coincide. As Moore's Law led to yet more powerful computers it became possible to represent three-dimensional behaviour (thereby

[5] Though the first nonlinear result had been published as a footnote by Veronis (1959), in the same year that the term *magnetoconvection* was introduced by Malkus (1959).

introducing small-scale dynamo action), and eventually, with the advent of massively parallel machines, to explore behaviour in a compressible layer (which is most relevant to a star). This in turn has led to studies of pattern formation and to group theoretical approaches.

Our aim is to focus on idealized model problems, governed by differential equations and boundary conditions that are precisely formulated. Any numerical models should be accurate, and all small-scale structures should be properly resolved. Repeating the calculations with different values of the control parameters is the key to probing the underlying structure of a problem. The patterns of behaviour that emerge can then be related to analytical models that display similar bifurcation properties. Moreover, this reductionist approach makes it possible to isolate and then to understand the key physical processes that are involved. Such a style of research contrasts sharply with that of direct numerical simulations, where the aim is to reproduce observed behaviour, for instance at the surface of a star, including all the effects – compressibility, ionization, chemical composition and radiative transport – that are involved. Any such computation obviously demands even more massive computer power. Ultimately, a full understanding will demand some synthesis of these two disparate approaches.

1.2 Outline of the book

We start, in the next chapter, with a brief introduction to magnetohydrodynamics, focusing first on kinematic behaviour (including flux concentration and flux expulsion), then on dynamical effects (including waves) and finally on kinematic dynamos. The following chapter covers linear stability theory, filling out the brief account above. After these preliminaries, we proceed to describe the mildly nonlinear regime, which is most amenable to analysis, in Chapter 4. Here we first consider weakly nonlinear behaviour, near the initial bifurcations, before going on to unfold the Takens–Bogdanov bifurcation. These analytical results are compared with truncated models and with two-dimensional (2D) numerical results. Next we go on to discuss the appearance of chaotic oscillations at a Shilnikov bifurcation, comparing numerical results with theoretical predictions. Finally, we consider an approximate treatment of highly nonlinear behaviour in the strong field regime.

Chapter 5 is devoted to 2D Boussinesq magnetoconvection and to the interpretation of numerical experiments in both Cartesian and axisymmetric geometries. An interesting feature is the behaviour of localized patterns, related to 'snaking' near the initial bifurcation. The effects of imposing an inclined magnetic field, leading to travelling waves, are also discussed. The following chapter progresses to 3D Boussinesq convection, with an emphasis on pattern selection. A numerical survey covers the transition from strong to weak field regimes and the appearance of small-scale dynamo action.

Going beyond all these considerations, we introduce the effects of rotation in Chapter 7, starting with the playoff between Lorentz and Coriolis forces in a plane layer. Next we discuss spherical systems and idealized dynamo models. After that we give a brief account of the geodynamo and planetary dynamos, and then go on to survey experimental approaches to dynamo action using liquid metals.

In Chapter 8 we move on to compressible convection, considering first the effects of breaking the up-down Boussinesq symmetry in a shallow layer. In 2D there are competitions between standing and travelling wave solutions, while oscillatory hexagons take over in 3D. Stratified compressible magnetoconvection in a deep layer leads to changing patterns, with flux separation and the formation of locally intense magnetic fields. The role of symmetries in pattern formation can be studied with the aid of equivariant bifurcation theory.

In the last chapter we proceed to summarize the properties of stellar dynamos, followed by some comments on MHD turbulence. Then we return to our initial motivation at the beginning of this chapter and discuss sunspots and photospheric magnetoconvection in the light of the knowledge we have gained. The travelling waves that appear when fields are inclined can be related to filamentary structures seen in sunspot penumbrae.

The book ends with four appendices: the first explains the Boussinesq and anelastic approximations that are used throughout the book, while the second provides a brief introduction to chaotic behaviour. Finally, we summarize the principal features of the closely related problem of double-diffusive convection and then apply them to magnetic buoyancy.

1.3 General references

Although this is the first monograph devoted exclusively to magnetoconvection, the development of the subject can be followed chronologically, starting with Chandrasekhar's (1961) massive tome and continuing with a series of reviews. Some of these are in the spirit of astrophysical fluid dynamics, while others are more firmly astrophysical. Proctor and Weiss (1982) summarized the state of knowledge at that time; aspects of subsequent progress have been covered by Hughes and Proctor (1988), Weiss (1991, 2003, 2012) and Proctor (1992, 2005). Astrophysical applications were surveyed by

Weiss (2001) and have also been discussed by Schüssler (2001, 2013), by Stein and Nordlund (2006) and, more recently, by Nordlund, Stein and Asplund (2009) and Stein (2012b). Many aspects of solar magnetoconvection also figure in the recent book by Thomas and Weiss (2008), while Glatzmaier (2013) has covered computational modelling of convection and magnetoconvection.

Among the many references on magnetohydrodynamics, we recommend the classic text by Cowling (1976a), the account by Roberts (1967), and relevant chapters of the books by Moffatt (1978), Parker (1979), Choudhuri (1998), Mestel (2012) and Priest (2014). Acheson (1990) and Thompson (2006a) offer good introductions to non-magnetic fluid dynamics and to astrophysical fluid dynamics, respectively.

2
Basic MHD

As a preliminary to embarking on the theory of magnetoconvection, it is necessary to provide an introduction to magnetohydrodynamics. Such a description comes in two parts: *kinematics*, which deals with the influence of motion on a magnetic field; and *dynamics*, which deals with the influence of the magnetic field on the velocity and other properties of the fluid. We start with kinematics, assuming that the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is known, and unaffected by the magnetic field, before proceeding to dynamics, and then return to discuss some relevant aspects of kinematic dynamo theory.

2.1 The induction equation

Any discussion of kinematic MHD must begin with the induction equation. We begin by introducing the magnetohydrodynamic approximation and then go on to consider behaviour of magnetic fields in perfectly conducting fluids, reserving the effects of finite conductivity for the next section.

Our starting point is Maxwell's equations for the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, electric field $\boldsymbol{E}(\boldsymbol{x}, t)$, electrostatic charge density $\rho_{\mathrm{E}}(\boldsymbol{x}, t)$ and current density $\boldsymbol{j}(\boldsymbol{x}, t)$. These take the form

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho_{\mathrm{E}}, \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}, \tag{2.3}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B} - \mu_0 \boldsymbol{j}. \tag{2.4}$$

Here c is the speed of light, and μ_0, ϵ_0 are the permeability and permittivity of vacuum, with $c^2 = (\epsilon_0\mu_0)^{-1}$. As is appropriate for solar plasmas and liquid metals, we shall ignore any dielectric or magnetic effects of media.

It is well known that the full Maxwell equations given above admit solutions in the form of electromagnetic waves, travelling at speed c. To see this, suppose that there is a vacuum, so that no currents flow and $\boldsymbol{j} = 0$: then $\boldsymbol{E}$ may be eliminated between (2.3) and (2.4) to give (making use of (2.2) and the vector identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) - \nabla^2$),

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = c^2 \nabla^2 \boldsymbol{B}, \tag{2.5}$$

which is the wave equation for each Cartesian component of $\boldsymbol{B}$. Now c is very large compared to any fluid velocities found in astrophysical bodies of interest, and it would pose immense difficulties to have to calculate the electromagnetic wave field as part of a convection calculation. If we suppose that typical phenomena occur on a timescale $\mathcal{T}$ and length scale $\mathcal{L}$, so that a typical velocity scale $\mathcal{U} \sim \mathcal{L}/\mathcal{T}$, and that $\Delta \equiv \mathcal{L}/c\mathcal{T} \ll 1$, then the relative sizes of $\boldsymbol{B}$ and $\boldsymbol{E}$ may be obtained from (2.3):

$$\frac{|\boldsymbol{B}|}{\mathcal{T}} \sim \frac{|\boldsymbol{E}|}{\mathcal{L}} \tag{2.6}$$

and so the ratio of the terms $c^{-2}\partial \boldsymbol{E}/\partial t$ (the *displacement current*) and $\boldsymbol{\nabla} \times \boldsymbol{B}$ in (2.4) may be estimated:

$$\frac{|c^{-2}\partial \boldsymbol{E}/\partial t|}{|\boldsymbol{\nabla} \times \boldsymbol{B}|} \sim \frac{\mathcal{L}|\boldsymbol{E}|}{c^2 \mathcal{T}|\mathcal{B}|} \sim \Delta^2 \ll 1. \tag{2.7}$$

Thus we can neglect the displacement current term and replace (2.4) by Ampère's 'pre-Maxwell' equation

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j}. \tag{2.8}$$

In order to close the system it is necessary to relate the current to the electric field. Unlike Maxwell's equations, such a relation depends on the nature of the fluid. For most normal purposes it is sufficient to adopt the simple relation known as *Ohm's Law*, which can be written

$$\boldsymbol{j} = \sigma \boldsymbol{E}', \tag{2.9}$$

where $\boldsymbol{E}'$ is the electric field *measured in the rest frame of the fluid*, and σ is the electrical conductivity of the fluid, often supposed uniform (though in astrophysical applications it may depend on temperature and density).[1]

[1] The conductivity σ is limited by collisions between electrons and positively charged ions. In a dilute plasma the electrons gyrate around field lines between collisions and a generalized form of Ohm's Law becomes appropriate (Cowling 1976a; Mestel 2012).