Introduction

ABOUT "REALISM"

A word on terminology may be useful at the outset, since it is pertinent to many of the papers in this collection, beginning with the very first. The label "realism" is used in two very different ways in two very different debates in contemporary philosophy of mathematics. For nominalists, "realism" means acceptance that there exist entities, for instance natural or rational or real numbers, that lack spatiotemporal location and do not causally interact with us. For neo-intuitionists, "realism" means acceptance that statements such as the twin primes conjecture may be true independently of any human ability to verify them. For the former the question of "realism" is ontological, for the latter it is semantico-epistemological. Since the concerns of nominalists and of neo-intuitionists are orthogonal, the double usage of "realism" affords ample opportunity for confusion.

The arch-nominalists Charles Chihara and Hartry Field, for instance, are anti-intuitionists and "realists" in the neo-intuitionists' sense. They do not believe there are any unverifiable truths about numbers, since they do not believe there are any numbers for unverifiable truths to be about. But they do believe that the facts about the possible production of linguistic expressions, or about proportionalities among physical quantities, which in their reconstructions replace facts about numbers, can obtain independently of any ability of ours to verify that they do so. Michael Dummett, the founder of neo-intuitionism, was an early and forceful anti-nominalist, and though he calls his position "anti-realism," he and his followers are "realists" in the nominalists' sense, accepting some though not all classical existence theorems, namely those that have constructive proofs, and agreeing that it is a category mistake to apply spatiotemporal or causal predicates to mathematical subjects.

On top of all this, even among those of us who are "realists" in both senses there are important differences. *Metaphysical* realists suppose, like

Galileo and Kepler and Descartes and other seventeenth-century worthies, that it is possible to get behind all human representations to a God's-eye view of ultimate reality as it is in itself. When *they* affirm that mathematical objects transcending space and time and causality exist, and mathematical truths transcending human verification obtain, they are affirming that such objects exist and such truths obtain *as part of ultimate metaphysical reality* (whatever that means). *Naturalist* realists, by contrast, affirm only (what even some self-described anti-realists concede) that the existence of such objects and obtaining of such truths is an implication or presupposition of science and scientifically informed common sense, while denying that philosophy has any access to exterior, ulterior, and superior sources of knowledge from which to "correct" science and scientifically informed common sense. The naturalized philosopher, in contrast to the alienated philosopher, is one who takes a stand as a citizen of the scientific community, and not a foreigner to it, and hence is prepared to reaffirm while doing philosophy whatever was affirmed while doing science, and to acknowledge its evident implications and presuppositions; but only the metaphysical philosopher takes the status of what is affirmed while doing philosophy to be a revelation of an ultimate metaphysical reality, rather than a human representation that is the way it is in part because a reality outside us is the way it is, and in part because we are the way we are.

My preferred label for my own position would now be "naturalism," but in the papers in this collection, beginning with the first, "realism" often appears. Were I rewriting, I might erase the R-word wherever it occurs; but as I said in the preface above, I do not believe in rewriting when reprinting, so while in date of composition the papers reproduced here span more than twenty years, still I have left even the oldest, apart from the correction of typographical errors, just as I wrote them. *Quod scripsi, scripsi*.

This collection begins with five items each pertinent in one way or another to nominalism and the problem of the existence of abstract entities. The term "realism" is used in an ontological sense in the first of these, "Numbers and ideas" (2003). This paper is a curtain-raiser, a lighter piece responding to certain professional mathematicians turned amateur philosophers who propose a cheap and easy solutions to the problem. According to their proposed compromise, numbers exist, but only "in the world of ideas." Since acceptance of this position would render most of the professional literature on the topic irrelevant, and since the amateurs often offer unflattering accounts of what they imagine to be the reasons why professionals do not accept their simple proposal, I thought it worthwhile to accept an invitation to try to state, for a general audience, our real reasons, which go back to Frege. The distinction insisted upon in this paper, between the kind of thing it makes sense to say about a number and the kind of thing it makes sense to say about a mental representation of a number (and the distinction, which exactly parallels that between the two senses of "history," between mathematics, the science, and mathematics, its subject matter) is presupposed throughout the papers to follow.

Some may wonder where my emphatic rejection of "idealism or conceptualism" in this paper leaves intuitionism. The short answer is that I leave intuitionism entirely out of account: I am concerned in this paper with descriptions of the mathematics we have, not prescriptions to replace it with something else. Intuitionism is orthogonal to nominalism, as I have said, and issues about it are set aside in the first part of this collection. I will add that, though I do not address the matter in the works reprinted here, my opinion is that Frege's anti-psychologistic and anti-mentalistic points raise some serious difficulties for Brouwer's original version of intuitionism, but no difficulties at all for Dummett's revised version. Neither opinion should be controversial. Dummett's producing a version immune to Fregean criticism can hardly surprise, given that the founder of neo-intuitionism is also the dean of contemporary Frege studies. That Brouwer's version, by contrast, faces serious problems was conceded even by so loyal a disciple as Heyting, and all the more so by contemporary neo-intuitionists.

AGAINST HERMENEUTIC AND REVOLUTIONARY NOMINALISM

"Why I am not a nominalist" (1983) represents my first attempt to articulate a certain complaint about nominalists, namely, their unclarity about the distinction between *is* and *ought*. It was this paper that first introduced a distinction between *hermeneutic* and *revolutionary* nominalism. The formulations a decade and a half later in *A Subject With No Object* (Burgess and Rosen, 1997) are, largely owing to my co-author Gideon Rosen, who among other things elaborated and refined the hermeneutic/revolutionary distinction, more careful on many points than those in this early paper. This piece, however, seemed to me to have the advantage of providing a more concise, if less precise, expression of key thoughts underlying that later book than can be found in any one place in the book itself. Inevitably I have over the years not merely elaborated but also revised (often under Rosen's influence) some of the views expressed in this early article.

First, the brief sketches of projects of Charles Chihara and Hartry Field in the appendix to the paper (which I include on the recommendation of an anonymous referee, having initially proposed dropping it in the reprinting) are in my present opinion more accurate as descriptions of aspirations than of achievements, and even then as descriptions only to a first approximation; moreover the later approach of Geoffrey Hellman is not discussed at all. My ultimate view of the technical side of the issue is given in full detail in the middle portions of *A Subject*, superseding several earlier technical papers.

Further, though I still see no serious linguistic evidence in favor of any hermeneutic nominalist conjectures, I no longer see the absence of such evidence as the main objection to them. For reasons that in essence go back to William Alston, such conjectures lack relevance *even if correct*. Even if we grant that "There are prime numbers greater than a million" does just mean, say, "There could have existed prime numerals greater than a million," the conclusion that should be drawn is that "Numbers exist" means "Numerals could have existed," and is therefore *true*, as anti-nominalists have always maintained, and not *false*, as nominalists have claimed. There is no threat at all to a naturalist version of anti-nominalism in such translations, though there might be to a metaphysical version. This line I first developed in a very belatedly published paper (Burgess 2002a) of which a condensed version was incorporated into *A Subject*.

Finally, I now recognize that there is a good deal more to be said for the position I labeled "instrumentalism" than I or almost anyone active in the field was prepared to grant back in the early 1980s when I wrote "Why I am not," or even in the middle 1990s, when I wrote my contributions to *A Subject*. The position in question is that of those philosophers who speak with the vulgar in everyday and scientific contexts, only to deny on entering the philosophy room that they meant what they said seriously. This view is now commonly labeled "fictionalism," and it deserves more discussion than it gets in either "Why I am not" or *A Subject*. It should be noted that while I originally opposed fictionalism (or instrumentalism) to both the revolutionary and hermeneutic positions, Rosen has correctly pointed out that fictionalism itself comes in a revolutionary version (this is the attitude philosophers *ought to* adopt) and a hermeneutic version (this is the attitude commonsense and scientific thinkers *already do* adopt). What I originally called the "hermeneutic" position should be called the "content-hermeneutic" position, and the hermeneutic version of fictionalism the "attitude-hermeneutic" position, in Rosen's refined terminology.

On two points my view has not changed at all over the past years. First, while nominalists would wish to blur what for Rosen and myself is a key distinction, and avoid taking a stand on whether they are giving a description of the mathematics we already have (hermeneutic) or a prescription for a new mathematics to replace it (revolutionary), gesturing towards a notion of "rational reconstruction" that would somehow manage to be neither the one nor the other, I did not think this notion had been adequately articulated when I first took up the issue of nominalism, and I have not found it adequately articulated in nominalist literature of the succeeding decades.

Second, as to the popular epistemological arguments to the effect that even if numbers or other objects "causally isolated" from us do exist, we cannot *know* that they do, I have not altered the opinions that I expressed in my papers Burgess (1989) and the belatedly published Burgess (1998b), and that Rosen expressed in his dissertation, and that the two of us jointly expressed in *A Subject*. The epistemological argument, according to which belief in abstract objects, even if conceded to be implicit in scientific and commonsense thought, and even if perhaps true – for the aim of going epistemological is precisely to avoid direct confrontation over the question of the *truth* of anti-nominalist existence claims – cannot constitute *knowledge*, surely is not intended as a Gettierological observation about the gap between justified true belief and what may properly be called *knowledge*. It follows that it must be an issue about *justification*; and here to the naturalized anti-nominalist the nominalist appears simply to be substituting some extra-, supra-, praeter-scientific *philosophical* standard of justification for the ordinary standards of justification employed by science and common sense: the naturalist anti-nominalist's answer to nominalist skepticism about mathematics is skepticism about philosophy's supposed access to such non-, un-, and anti-scientific standards of justification.

AGAINST FICTIONALIST NOMINALISM

Returning to the issue of fictionalism, in our subsequent work Rosen and I have generally dealt with it separately and in our own ways. A chapter bearing the names of Rosen and myself, "Nominalism reconsidered," does appear in Stuart Shapiro's *Handbook of Philosophy of Mathematics and Logic* (2005), and it is a sequel to our book adding coverage of fictionalist nominalism, with special reference to the version vigorously advocated over the past several years by Steve Yablo; but this chapter is substantially Rosen's work, my contributions being mainly editorial.

My own efforts to address a fictionalist position are to be found rather in "Mathematics and *Bleak House*," which revisits, in a sympathetic spirit, Rudolf Carnap's ideas on the status of ontological questions and nominalist theses. Neo-Carnapianism is on the rise, and I am happy to be associated with it, though like any other neo-Carnapian I have my differences with my fellow neo-Carnapians. "Quine, analyticity, and philosophy of mathematics" can be read as a sequel to the *Bleak House* paper (it was written much later, though owing to various accidents both came out in the same year, 2004). It revisits the famous exchange between Carnap and Quine on ontology, again in a spirit sympathetic to Carnap.

Carnap thought there was a separation to be made between analytic questions about what is the content of a concept such as that of number, and pragmatic questions about why we accept such a concept for use in scientific theorizing and commonsense thought. Quine denied there was in theory any sharp separation to be made. I argue that there is in practice at least a fuzzy one. I also argue that Quine had better acknowledge as much if he is to be able to make any reply to a serious criticism of Charles Parsons. The criticism is that Quine's holist conception of the justification of mathematics – it counts as a branch of science rather than imaginative literature because of its contribution to other sciences – cannot do justice to the *obviousness* of elementary arithmetic.

Though placed in the first half of this volume along with papers about nominalism, the Quine paper can equally well be read more or less independently as a paper in philosophy of language and theory of knowledge about the notion of analyticity, one that just happens to use mathematics and logic as sources of examples. The placement of this paper, and more generally the division of the collection into two parts, should not be taken too seriously.

As any neo-Carnapian will tell you, though Carnap was certainly an anti-nominalist, his position is perhaps better characterized as generally anti-*ontological* rather than specifically anti-*nominalist*. My own general anti-ontologism became finally, fully, and emphatically explicit in "Being explained away" (2005), my farewell to the issue of nominalism. In this retrospective (written for an audience of undergraduate philosophy concentrators) I distinguish what I call scientific *ontics*, a glorified taxonomy of the entities recognized by science, from what I call philosophical *ontosophy*, an impossible attempt to get behind scientific representations to a God's-eye view, and catalogue the metaphysically ultimate furniture of the universe. The error of the nominalists consists, in my opinion, not in ontosophical anti-realism about the abstract, but in ontosophical realism about the concrete – more briefly, the error is simply going in for ontosophy and not resting content with ontics.

In taking leave of the issue of nominalism, I should reiterate the point made briefly at the end of *A Subject*, that from a naturalist point of view there is a great deal to be learned from the projects of Field, Chihara, Hellman, and others. Naturalists, I have said, hold that there is no possibility of separating completely the contributions from the world and the contributions from us in shaping our theories of the world. At most we can get a hint by considering how the theories of creatures like us in a world unlike ours, or the theories of creatures unlike us in a world like ours, might differ from our own theories. The nominalist reconstruals or reconstructions, though implausible when read as hermeneutic, as accounts of the meaning of our theories, and unattractive when read as revolutionary, as rivals competing for our acceptance with those theories, do give a hint of what the theories of creatures unlike us might be like.

Another hint is provided by those monist philosophers who have reconstrued what appear to be predicates applying to various objects as predicates applying to a single subject, the Absolute, with the phrases that seem to refer to the various objects being reconstrued as various adverbial modifiers. Thus "Jack sings and Jill dances" becomes "The Absolute sings jackishly and dances jillishly," while "Someone sings and someone else dances" becomes "The Absolute sings somehow and dances otherhow." What is specifically sketched in "Being explained away" is how this kind of reconstrual can be systematically extended, at least as far as first-order regimentation of discourse can be extended. Of course it is not to be expected that we can fully imagine what it would be like to be an intelligent creature who habitually thought in such alien terms, any more that we can fully imagine what it would be like to be a bat. Nor insofar as we are capable of partially imagining what is not wholly imaginable are formal studies the only aid to imagination. The kind of fiction that stands to metaphysics as science fiction stands to physics – the example I cite in the paper is Borges – may give greater assistance.

FOUNDATIONS OF MATHEMATICS: SET THEORY

As long as mathematicians adhere to the ideal of rigorous proof from explicit axioms, they will face decisions as to which proposed axioms to start from, and which methods of proof to admit. What is conventionally known as "foundations of mathematics" is simply the technical study, using the tools of modern logic, of the effects of different choices. Work in foundations emphatically does not imply commitment to a "foundationalist" philosophical position, or for that matter to *any* philosophical position. In Burgess (1993) I nevertheless argued that work in foundations can be *relevant* to philosophy, and tried to explain how. I will not attempt to summarize the explanation here, except to give this hint: most of the interesting choices of axioms, especially those that are more restrictive rather than the orthodox choice of something like the axioms of Zermelo–Frankel set theory, were originally inspired by positions in the philosophy of mathematics (finitism, constructivism, predicativism, and others). Foundational work helps us appreciate what is at stake in the choice among those restrictive philosophies, and between them and classical orthodoxy.

While the early papers in the first part of this collection are predominantly though not exclusively critical, and the middle papers a mix of critical and positive – I would say "constructive," except that this word has a special meaning in philosophy of mathematics – the last two are, like the bulk of my more technical work, predominantly though not exclusively positive. Though they do not endorse as ultimately correct, they present as deserving of serious and sustained attention three novel approaches to foundations of mathematics, very different in appearance from each other, but not necessarily incompatible.

To the extent that there is an agreed foundation or framework for contemporary pure mathematics, it is provided by something like the Zermelo–Frankel system of axiomatic set theory, in the version including the axiom of choice (ZFC). "*E pluribus unum*" (2004) attempts to combine two insights, one due to Boolos, the other to Paul Bernays, to achieve an improved framework.

The idea taken from Boolos is that plural quantification on the order of "there are some things, the *u*s, such that . . ." is a more primitive notion than singular quantification of the type "there is a set or class U of things such that . . ." and that Cantor's transition from the former to the latter was a genuine conceptual innovation, not a mere uncovering of a commitment to set- or class-like entities that had been implicit in ordinary plural talk all along.

Boolos himself had applied this idea to set theory, to suggest, not improved axioms, but an improved formulation of the existing axioms. For there is a well-known awkwardness in the formulation of ZFC, in that two of its most important principles appear not as axioms but as *schemes*, or rules to the effect that all sentences of a certain form are to count as axioms. For instance, *separation* takes the form

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \ \& \ \phi(z))$$

wherein ϕ may be any formula. Needless to say, no one becomes convinced of the correctness of ZFC by becoming convinced separately of each of infinitely many instances of the separation scheme. But the language of ZFC provides no means of formulating the underlying *single*, *unified* principle. One proposed solution to this difficulty has been to recognize collections of a kind called *classes* that are set-like while somehow failing to be sets. With capital letters ranging over such entities, and with "$z \in U$" written "Uz" to emphasize that the relation of class membership is a kind of belonging that is like set-elementhood and yet somehow fails to be set-elementhood, the separation scheme can be reduce to a single axiom, thus:

$$\forall U \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \ \&\ Uz).$$

But notion of class brings with it difficulties of its own, leaving many hesitant to admit these alleged entities.

The suggestion of Boolos (in my own notation) was to replace singular quantification $\forall U$ or "for any class U of sets . . ." over classes by plural quantification $\forall\forall uu$ or "for any sets, the u's . . ." and Uz or "z is a member of U" by $z \propto uu$ or "z is one of the u's," thus yielding a formulation in which the only objects quantified over are sets:

$$\forall\forall uu \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \ \& z \propto uu).$$

One may even take a further step and make the notion $x \equiv uu$ or "x is the set of the u's" primitive, with the notion $y \in x$ or "y is an element of x" being defined in terms of it, as $\exists\exists uu(x\, x \equiv uu \ \&\ y \propto uu)$ or "there are some things that x is the set of, and y is one of them." Such a step was actually taken in a paper by Stephen Pollard (1996) some years before my own, of which I only belated became aware, along with Shapiro (1987) and Rayo and Uzquiano (1999).

The idea taken from Bernays was that an approach incorporating a so-called reflection principle can provide a simpler axiomatization than the standard approach to motivating the axioms of ZFC, and permit the derivation of some further so-called large-cardinal principles that are widely accepted by set theorists, though they go beyond ZFC. The original Bernays approach had the disadvantage of involving "classes" over and above sets, and of requiring a somewhat artificial technical condition in the formulation of the reflection principle. Boolos's plural logic was subject to the objection that, like any version or variant of second-order logic, it lacks a complete axiomatization. I aim to show how the combination of Boolos with Bernays neutralizes these objections.

FOUNDATIONS OF MATHEMATICS: LOGICISM

"Logicism: a new look" (previously unpublished) provides a concise, semi-popular introduction to two alternative approaches to foundations each of which I have examined more fully and technically elsewhere. Each represents a version of the old idea of *logicism*, according to which mathematics is ultimately but a branch of logic. Computational facts such as $2 + 2 = 4$, on this view, become abbreviations for logical facts; in this case, the fact that if there exists an F and another and no more, and a G and another and no more, and nothing is both an F and a G, and something is an H if and only if it is either an F or a G, then there exists an H and another and yet another and still yet another, but no more.

One new idea derives from Richard Heck. Frege, the founder of modern logic and modern logicism proposed to develop arithmetic in a grand system of logic of his devising. That system is, in modern notation and to a first approximation, a form of second-order logic, with axioms of *comprehension* and *extensionality*,

$$\exists X \forall x(Xx \leftrightarrow \phi(x))$$
$$\forall X \forall Y(\forall z(Xz \leftrightarrow Yz) \rightarrow (\phi(X) \leftrightarrow \phi(Y)))$$

supplemented by an axiom to the effect that to each second-order entity X there is associated a first-order entity X^* in such a way that we have

$$\forall X \forall Y(X^* = Y^* \leftrightarrow \forall z(Xz \leftrightarrow Yz)).$$

Russell showed that a paradox arises in this system, and also introduced the idea of imposing a restriction of *predicativity* on the comprehension axiom, assuming it only for formulas $\phi(x)$ without bound class variables. Russell proposed a great many other changes, and his overall system diverged greatly from Frege's. Heck was the first to consider closely what would happen if one made *only* the one change just described, and he showed that the resulting system, though weak (and in particular consistent) is strong enough for the minimal arithmetic embodied in the system known in the literature as Q to be developed in it. So a bare minimum of mathematics can be developed on a predicative logicist basis in the manner of Frege. (More technical details as to what can be accomplished along these lines are provided in my book *Fixing Frege* (Burgess 2005b). Since the book and the paper were written there have been important advances by Mihai Ganea and Albert Visser.)