DIFFERENTIAL GEOMETRY AND LIE GROUPS FOR PHYSICISTS

Differential geometry plays an increasingly important role in modern theoretical physics and applied mathematics. This textbook gives an introduction to geometrical topics useful in theoretical physics and applied mathematics, including manifolds, tensor fields, differential forms, connections, symplectic geometry, actions of Lie groups, bundles and spinors.

Having written it in an informal style, the author gives a strong emphasis on developing the understanding of the general theory through more than 1000 simple exercises, with complete solutions or detailed hints. The book will prepare readers for studying modern treatments of Lagrangian and Hamiltonian mechanics, electromagnetism, gauge fields, relativity and gravitation.

Differential Geometry and Lie Groups for Physicists is well suited for courses in physics, mathematics and engineering for advanced undergraduate or graduate students, and can also be used for active self-study. The required mathematical background knowledge does not go beyond the level of standard introductory undergraduate mathematics courses.

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Preface

This is an introductory text dealing with a part of mathematics: modern differential geometry and the theory of Lie groups. It is written from the perspective of and mainly for the needs of physicists. The orientation on physics makes itself felt in the choice of material, in the way it is presented (e.g. with no use of a definition–theorem–proof scheme), as well as in the content of exercises (often they are closely related to physics).

Its potential readership does not, however, consist of physicists alone. Since the book is about mathematics, and since physics has served for a fairly long time as a rich source of inspiration for mathematics, it might be useful for the mathematical community as well. More generally, it is suitable for anybody who has some (rather modest) preliminary background knowledge (to be specified in a while) and who desires to become familiar in a comprehensible way with this interesting, important and living subject, which penetrates increasingly into various branches of modern theoretical physics, "pure" mathematics itself, as well as into its numerous applications.

So, what is the minimal background knowledge necessary for a meaningful study of this book? As mentioned above, the demands are fairly modest. Indeed, the required mathematical background knowledge does not go beyond what should be familiar from standard introductory undergraduate mathematics courses taken by physics or even engineering majors. This, in particular, includes some calculus as well as linear algebra (the reader should be familiar with things like partial derivatives, several variables Taylor expansion, multiple Riemann integral, linear maps versus matrices, bases and subspaces of a linear space and so on). Some experience in writing and solving simple systems of ordinary differential equations, as well as a clear understanding of what is actually behind this activity, is highly desirable. Necessary basics in algebra in the form used in the main text are concisely summarized in Appendix A at the end of the book, enabling the reader to fill particular gaps "on the run," too.

The book is intentionally written in a form which makes it possible to be fully grasped also by a self-taught person – anybody who is attracted by tensor and spinor fields or by fiber bundles, who would like to learn how differential forms are differentiated and integrated, who wants to see how symmetries are related to Lie groups and algebras as well as to their representations, what is curvature and torsion, why symplectic geometry is useful in Lagrangian and Hamiltonian mechanics, in what sense connections and gauge fields realize the same idea, how Noetherian currents emerge and how they are related to conservation laws, etc.

Clearly, it is highly advantageous, as the scope of the book indicates, to be familiar (at least superficially) with the relevant parts of physics on which the applications of various techniques are illustrated. However, one may derive profit from the book (in terms of geometry alone) even with no background from physics. If we have never seen, say, Maxwell's equations and we are not aware at all of their role in physics, then although we will not be able to understand *why* such attention is paid to them, nevertheless we will understand perfectly *what* we do with these equations here from the technical point of view. We will see how these partial differential equations may be reformulated in terms of differential forms, what the action integral looks like in this particular case, how conservation laws may be derived from it by means of the energy–momentum tensor and so on. And if we find it interesting, we may hopefully also learn some "traditional" material on electrodynamics later.

If we, in like manner, know nothing about general relativity, then although we will not understand from where the concept of a "curved" space-time endowed with a metric tensor emerged, still we will learn the basics of what space-time is from a geometrical point of view and what is generally done there. We will not penetrate into the physical heart of the Einstein equations for the gravitational field, we will see, however, their formal structure and we will learn some simple, though at the same time powerful, techniques for routine manipulations with these equations. Mastering this machinery then greatly facilitates grasping the physical side of the theory, if later we were to read something written about general relativity from the physical perspective.

The key qualification asked of the future reader is a real interest in learning the subject treated in the book not only in a Platonic way (say, for the sake of an intellectual conversation at a party) but rather at a working level. Needless to say, one then has to accept a natural consequence: it is not possible to achieve this objective by a passive reading of a "noble science" alone. On the contrary, a fairly large amount of "dirty" self-activity is needed (an ideal potential reader should be *pleased* by reading this fact), inevitably combined with due investment of time. The formal organization of the book strongly promotes this method of study.

A specific feature of the book is its strong emphasis on developing the general theory through a large number of simple exercises (more than a thousand of them), in which the reader analyzes "in a hands-on fashion" various details of a "theory" as well as plenty of concrete examples (the proof of the pudding is in the eating). This style is highly appreciated, according to my teaching experience, by many students.

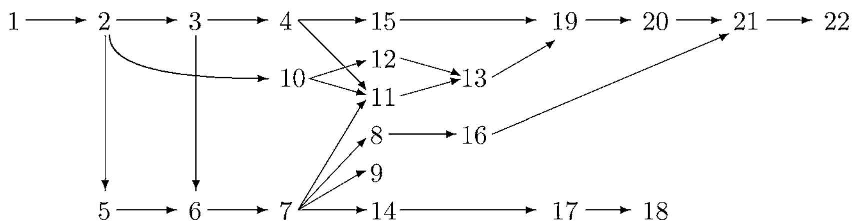
The beginning of an exercise is indicated by a box containing its number (as an example, [14.4.3] denotes the third exercise in Section 4, Chapter 14), the end of the exercise is marked by a square □. The majority of exercises (around 900) are endowed with a hint (often quite detailed) and some of them, around 50, with a worked solution. The symbol • marks the beginning of "text," which is not an exercise (a "theory" or a comment to exercises). Starred sections (like 12.6*) as well as starred exercises may be omitted at the first reading (they may be regarded as a complement to the "hard core" of the book; actually they need not be harder but more specific material is often treated there).

This book contains a fairly large amount of material, so that a few words might be useful on how to read it efficiently. There are several ways to proceed, depending on what we actually need and how much time and effort we are willing to devote to the study.

The basic way, which we recommend the most, consists in reading systematically from cover to cover and solving (nearly) all the problems step by step. This is the way in which we may make full use of the text. The subject may be understood in sufficient broadness, with a lot of interrelations and applications. This needs, however, enough motivation and patience.

If we lack either, we may proceed differently. Namely, we will solve in detail only those problems which we, for some reason, regard as particularly interesting or from which we crucially need the result. Proceeding in this way, it may happen here and there that we will not be able to solve some problem; we are lacking some vital link (knowledge or possibly a skill) treated in the material being omitted. If we are able to locate the missing link (the numbers of useful previous exercises, mentioned in hints, might help in doing so), we simply fill this gap at the relevant point.

Yet more quickly will proceed a reader who decides to restrict their study to a particular direction of interest and who is interested in the rest of the book only to the extent that it is important for his or her preferred direction. As an aid to such a reader we present here a scheme showing the logical dependence of the chapters:



(The scheme does not represent the dependence completely; several sections, short parts or even individual exercises would require the drawing of additional arrows, making the scheme then, however, virtually worthless.)

To be more explicit, one could mention the following possible particular directions of interest.

1. The geometry needed for the fundamentals of **general relativity** (**covariant derivatives**, **curvature tensor**, **geodesics**, etc.).

   One should follow the line 1 → 2 → 3 → 4 → 15 (similar material goes well with advanced **continuum mechanics**). If we want to master working with forms, too (to grasp, as an example, Section 15.6, dealing with the computation of the Riemann tensor in terms of Cartan's structure equations, or Section 16.5 on Einstein's equations and their derivation from an action integral), we have to add Chapters 5–7.
2. **Elementary** theory of **Lie groups** and their **representations** ("(differential) geometry-free mini-course").

   The route might contain the chapters (or only the explicitly mentioned sections of some of them) 1 → 2.4 → 10 → 11.7 → 12 → 13.1–13.3.

3. **Hamiltonian** mechanics and **symplectic manifolds**.
   The minimal itinerary contains Chapters 1 → 2 → 3 → beginning of 4 → 5 → 6 → 7 → 14. Its extension (the formulation of Lagrangian and Hamiltonian mechanics on the fiber bundles TM and T^*M respectively) takes place in Chapters 17 and 18. If we have the ambition to follow the more advanced sections on symmetries (Sections 14.5–14.7 and 18.4), we need to understand the geometry on Lie groups and the actions of Lie groups on manifolds (Chapters 11–13).
4. Basics of working with **differential forms**.
   The route could be 1 → 2 → 3 → beginning of 4 → 5 → 6 → 7 → 8 → 9, or perhaps adding the beginning of Chapter 16.

This book stems from (and in turn covers) several courses I started to give roughly 15 years ago for theoretical physics students at the Faculty of Mathematics and Physics in Bratislava. It has been, however, extended (for the convenience of those smart students who are interested in a broader scope on the subject) as well as polished a bit (although its presentation often still resembles more the style of informal lectures than that of a dry "noble-science monograph"). In order to mention an example of **how the book may be used by a teacher**, let me briefly note what **four** particular formal **courses** are covered by the book. The first, fairly broad one, is compulsory and it corresponds roughly to (parts of) Chapters 1–9 and 14–16. Thus it is devoted to the essentials of general differential geometry and an outline of its principal applications. The other three courses are optional and they treat more specific parts of the subject. Namely, (elementary) Lie groups and algebras and their representations (it reproduces more or less the "particular direction of interest" number 2, mentioned above), geometrical methods in classical mechanics (the rest of Chapter 14 and Chapters 17 and 18) and connections and gauge fields (Chapters 19–21).

I have benefited from numerous discussions about geometry in physics with colleagues from the Department of Theoretical Physics, in particular with Paľo Ševera and Vlado Balek.

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Finally, I wish to thank my wife, L'ubka, and my children, Stanko, Mirko and Danka, for the considerable amount of patience displayed during the years it took me to write this book.

I tried hard to make *Differential Geometry and Lie Groups for Physicists* error-free, but spotting mistakes in one's own writing can be difficult in a book-length work. If you notice any errors in the book or have suggestions for improvements, please let me know (fecko@fmph.uniba.sk). Errors reported to me (or found by myself) will be listed at my web page

http://sophia.dtp.fmph.uniba.sk/~fecko

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