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Introduction

In physics every now and then one needs something to differentiate or integrate. This is the reason why a novice in the field is simultaneously initiated into the secrets of differential and integral calculus.

One starts with functions of a single variable, then several variables occur. Multiple integrals and partial derivatives arrive on the scene, and one calculates plenty of them on the drilling ground in order to survive in the battlefield.

However, if we scan carefully the structure of expressions containing partial derivatives in real physics formulas, we observe that some combinations are found fairly often, but other ones practically never occur. If, for example, the frequency of the expressions

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \quad \text{and} \quad \frac{\partial^3 f}{\partial x^3}+\frac{\partial^2 f}{\partial y \partial z}+4\frac{\partial f}{\partial z}$$

is compared, we come to the result that the first one (Laplace operator applied to a function f) is met very often, while the second one may be found only in problem books on calculus (where it occurs for didactic reasons alone). Combinations which do enter real physics books, result, as a rule, from a computation which realizes some *visual local geometrical* conception corresponding to the problem under consideration (like a phenomenological description of diffusion of matter in a homogeneous medium). These very conceptions constitute the subject of a systematic study of *local differential geometry*. In accordance with physical experience it is observed there that there is a fairly small number of truly interesting (and, consequently, frequently met) operations to be studied in detail (which is good news – they can be mastered in a reasonably short time).

We know from our experience in general physics that the same situation may be treated using *various kinds of coordinates* (Cartesian, spherical polar, cylindrical, etc.) and it is clear from the context that the *result* certainly *does not depend* on the choice of coordinates (which is, however, far from being true concerning the *sweat involved* in the computation – the very reason a careful choice of coordinates is a part of wise strategy in solving problems). Thus, both objects and operations on them are independent of the choice of coordinates used to describe them. It should be not surprising, then, that in a properly built formalism a great deal of the work may be performed using *no coordinates* whatsoever (just what part of the computation it is depends both on the problem and on the level of mastery of a particular

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user). There are several advantages which should be mentioned in favor of these "abstract" (coordinate-free) computations. They tend to be considerably shorter and more transparent, making repeated checking, as an example, much easier, individual steps may be better understood visually and so on. Consider, in order to illustrate this fact, the following equations:

$$\begin{aligned}
\mathcal{L}_\xi g = 0 \quad &\leftrightarrow \quad \xi^k g_{ij,k} + \xi^k{}_{,i} g_{kj} + \xi^k{}_{,j} g_{ik} = 0 \\
\nabla_{\dot\gamma}\dot\gamma = 0 \quad &\leftrightarrow \quad \ddot x^i + \Gamma^i_{jk}\dot x^j \dot x^k = 0 \\
\nabla g = 0 \quad &\leftrightarrow \quad g_{ij,k} - \Gamma_{ijk} - \Gamma_{jik} = 0
\end{aligned}$$

We will learn step by step in this book that the pairs of equations standing on the left and on the right side of the same line always tell us *just the same*: the expression on the right may be regarded as being obtained from that on the left by expressing it in (arbitrary) coordinates.

(The first line represents *Killing equations*; they tell us that the Lie derivative of g along ξ vanishes, i.e. that the metric tensor g has a *symmetry* given by a vector field ξ. The second one defines particular curves called geodesics, representing uniform motion in a straight line (= its acceleration vanishes). The third one encodes the fact that a linear connection is metric; it says that a scalar product of vectors remains unchanged under parallel translation.)

In spite of the highly efficient way of writing of the coordinate versions of the equations (partial derivatives via commas and the summation convention – we sum on indices repeating twice (dummy indices) omitting the $\sum$ sign), it is clear that they can hardly compete with the left side's brevity. Thus if we will be able to *reliably manipulate* the objects occurring on the left, we gain an ability to manipulate (indirectly) fairly complicated expressions containing partial derivatives, always keeping under control what we *actually* do.

At the introductory level calculus used to be developed in Cartesian space $\mathbb{R}^n$ or in open domains in $\mathbb{R}^n$. In numerous cases, however, we apply the calculus in spaces which *are not* open domains in $\mathbb{R}^n$, although they are "very close" to them.

In analytical mechanics, as an example, we study the motion of pendulums by solving (differential) Lagrange equations for coordinates introduced in the pendulum's configuration spaces, regarded as functions of time. These configuration spaces are not, however, open domains in $\mathbb{R}^n$. Take a simple pendulum swinging in a plane. Its configuration space is clearly a *circle* S^1. Although this is a one-dimensional space, it is intuitively clear (and one may prove) that it is essentially *different* from (an open set in) $\mathbb{R}^1$. Similarly the configuration space of a spherical pendulum happens to be the two-dimensional sphere S^2, which differs from (an open set in) $\mathbb{R}^2$.

Notice, however, that a sufficiently *small neighborhood* of an arbitrary point on S^1 or S^2 is practically indistinguishable from a sufficiently small neighborhood of an arbitrary point in $\mathbb{R}^1$ or $\mathbb{R}^2$ respectively; they are in a sense "locally equal," the difference being "only global." Various applications of mathematical analysis (including those in physics) thus strongly motivate its extension to more general spaces than those which are simple open domains in $\mathbb{R}^n$.

Such more general spaces are provided by *smooth manifolds*. Loosely speaking they are spaces which a *short-sighted observer* regards as $\mathbb{R}^n$ (for suitable n), but globally

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("topologically," when a pair of spectacles are found at last) their structure may differ profoundly from $\mathbb{R}^n$.

We can regard as an enjoyable bonus the fact that the formalism, which will be developed in order to perform coordinate-free computations, happens to be at the same time (free of charge) well suited to treating *global* geometrical problems, too, i.e. we may study the objects and operations on them, being well defined *on the manifold* as a whole. Therefore, we speak sometimes about *global analysis*, or the analysis on manifolds. All the above-mentioned equations $\mathcal{L}_\xi g = 0$, $\nabla_{\dot\gamma}\dot\gamma = 0$ and $\nabla g = 0$ represent, to give an example, equations on manifolds and their solutions may be defined as objects living on manifolds, too.

The key concept of a manifold itself will be introduced in Chapter 1. The exposition is mainly at the intuitive level. A good deal of material treated in detail in mathematical texts on differential *topology* will only be mentioned in a fairly informative way or will even be omitted completely. The aim of this introductory chapter is to provide the reader with a minimal amount of material which is necessary to grasp (fully, already at the working level) the main topic of the book, which is differential *geometry* on manifolds.

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1

The concept of a manifold

• The purpose of this chapter is to introduce the concept of a smooth manifold, including the ABCs of the technical side of its description. The main idea is to regard a manifold as being "glued-up" from *several* pieces, all of them being very simple (open domains in $\mathbb{R}^n$). The notions of a *chart* (local coordinates) and an *atlas* serve as essential formal tools in achieving this objective.

In the introductory section we also briefly touch upon the concept of a topological space, but for the level of knowledge of manifold theory we need in this book it will not be used later in any non-trivial way.

(From the didactic point of view our exposition leans heavily on recent scientific knowledge, for the most part on ethnological studies of Amazon Basin Indians. The studies proved convincingly that even those prodigious virtuosos of the art of survival within wild jungle conditions make do with only intuitive knowledge of smooth manifolds and the medicine-men were the only members within the tribe who were (here and there) able to declaim some formal definitions. The fact, to give an example, that the topological space underlying the smooth manifold should be *Hausdorff* was observed to be told to a tribe member just before death and as eyewitnesses reported, when the medicine-man embarked on analyzing examples of *non-Hausdorff* spaces, the horrified individual preferred to leave his or her soul to God's hands as soon as possible.)

1.1 Topology and continuous maps

• *Topology* is a useful structure a set may be endowed with (and at the same time the branch of mathematics dealing with these things). It enables one to speak about continuous maps. Namely, in order to introduce a topology on a set X, one has to choose a system $\{\tau\}$ of subsets τ of the set X, such that

1. $\emptyset \in \{\tau\}, \ X \in \{\tau\}$;
2. the union (of an arbitrary number) of elements from $\{\tau\}$ is again in $\{\tau\}$;
3. the intersection of a finite number of elements from $\{\tau\}$ is again in $\{\tau\}$.

(So that the system necessarily contains the empty set as well as the set X itself, and is closed with respect to arbitrary unions and finite intersections.) The elements of $\{\tau\}$ are

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called *open sets* and the pair $(X, \{\tau\})$ is a *topological space*. Given two topological spaces $(X, \{\tau\})$ and $(Y, \{\sigma\})$, a map

$$f : X \to Y$$

is said to be *continuous* if $f^{-1}(A) \in \{\tau\}$ for any $A \in \{\sigma\}$, that is to say if the inverse image[1] of any open set is again an open set.[2] Moreover, if the map f happens to be bijective and f^{-1} is continuous as well, f is called a *homeomorphism* (topological map); X and Y are then said to be homeomorphic.

$\boxed{1.1.1}$ Verify that the "weakest" (coarsest) possible topology on a set X is given by the *trivial topology*, where $\emptyset$ and X represent the only open sets available, whereas the "strongest" (finest) topology is the *discrete topology*, where *every* subset is open (in particular, this is also true for every point $x \in X$); all other topologies reside "somewhere between" these two extreme possibilities. □

$\boxed{1.1.2}$ Let $\{\tau\}_0, \{\tau\}_1$ be the trivial and the discrete topology respectively (1.1.1). Describe all *continuous* maps

$$f : (X, \{\tau\}_a) \to (Y, \{\tau\}_b) \qquad a, b \in \{0, 1\}$$

realizing thus that continuity of a map depends, in general, on the choice of topologies both on X and Y. (For $a = 1$ (b arbitrary) and for $a = 0 = b$ *all* maps are continuous; for $a = 0, b = 1$ the only continuous maps are *constant* maps ($x \mapsto y_0$, the same for all x).) □

$\boxed{1.1.3}$ Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

f, g being continuous. Show that the composition map

$$g \circ f : X \to Z$$

is continuous, too. □

$\boxed{1.1.4}$ Check that the notion of homeomorphism introduces an equivalence relation among topological spaces (reflexivity, symmetry and transitivity are to be verified). □

• The reader may find it helpful to visualize homeomorphic spaces as being made of rubber; Y can then be obtained from X by means of a deformation alone (neither cutting nor gluing are allowed). Example: a circle, a square and a triangle are all homeomorphic, the figure-of-eight symbol is not homeomorphic to the circle (provided that the intersection in the middle of it is regarded as a *single* point).[3]

[1] Recall that f^{-1} does not mean the inverse map here (this may not exist at all); $f^{-1}(A)$ denotes the collection of all elements in X which f sends into A, i.e. the *inverse image* of the set A.
[2] In elementary calculus continuity used to be defined in terms of *distance*; this turns out to be a particular case of the above definition (the distance induces a topology, to be mentioned later).
[3] *Differential Topology* by A. H. Wallace can be recommended as a nice introductory text about topology (see the Bibliography for details).

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One usually restricts oneself (for purely technical reasons, in order not to allow for manifolds of some fairly complicated objects that we do not want to be concerned with) to *Hausdorff* topological spaces. In these spaces (by definition), given any two points x, y, there exist non-intersecting neighborhoods of them (open sets A, B, such that $x \in A, y \in B, A \cap B = \emptyset$); one can thus *separate* any two points by means of open sets. From now on Hausdorff spaces will be understood automatically when speaking about topological spaces.

The fact that the *Cartesian space* $\mathbb{R}^n$ (ordered n-tuples of real numbers) represents a topological space (where open sets coincide with those used in the elementary calculus of n real variables) will be important in what follows.

1.1.5 Let $d(x, y)$ be the standard Euclidean distance between two points $x, y \in \mathbb{R}^n$, i.e.

$$d^2(x, y) := (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$$

and let

$$D(a, r) := \{x \in \mathbb{R}^n, d(x, a) < r\}$$

(open ball $\equiv$ disk, centered at a, the radius being r). A set $A \in \mathbb{R}^n$ is open if for any point $x \in A$ there exists an open ball centered at x which lies entirely in A. Check that this definition of an open set meets the axioms of a topological space. This topology is called the *standard topology in* $\mathbb{R}^n$. □

1.2 Classes of smoothness of maps of Cartesian spaces

- Let A be an open set in $\mathbb{R}^n[x^1, \dots, x^n]$ and

$$f : A \to \mathbb{R}^m[y^1, \dots, y^m]$$

This means that we are given m functions of n variables

$$y^a = y^a(x^1, \dots, x^n) \qquad a = 1, \dots, m$$

If all partial derivatives up to order k exist and are continuous, then f is called a map of *class* C^k. In particular, it is called *continuous* ($k = 0$), *differentiable* ($k = 1$), *smooth* ($k = \infty$) and (real) *analytic* (if for all $x \in A$ the Taylor series of $y^a(x)$ converges to the function $y^a(x)$ itself: $k = \omega$). In general, there clearly holds

$$C^0(A, \mathbb{R}^m) \supset C^1(A, \mathbb{R}^m) \supset \cdots \supset C^\infty(A, \mathbb{R}^m) \supset C^\omega(A, \mathbb{R}^m).$$

Far less trivial is the fact that not a single inclusion is in fact equality.

1.2.1 Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$\begin{aligned} f(x) &= e^{-\frac{1}{x}} & x &> 0 \\ f(x) &= 0 & x &\leq 0 \end{aligned}$$

Use this function to prove that in general $C^\omega(A, \mathbb{R}^m) \neq C^\infty(A, \mathbb{R}^m)$.

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Hint: show, that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \dots$ (so that the Taylor series in the neighborhood of $x = 0$ gives a function which vanishes *for positive* x, too). □

1.3 Smooth structure, smooth manifold

- A tourist map may be regarded as a true *map* (in the mathematical sense of the word)

$$\varphi : \mathrm{TD} \to \mathrm{SP}$$

where TD is a tourist district and SP is a sheet of paper. If the sheet of paper happens to be in fact in a square paper exercise book, we have another map

$$\chi : \mathrm{SP} \to \mathbb{R}^2[x^1, x^2]$$

and their composition results finally in

$$\psi : \mathrm{TD} \to \mathbb{R}^2 \qquad \psi \equiv \chi \circ \varphi$$

For a good map ψ should be a bijection and this makes it possible to assign a pair of real numbers – its coordinates – to any point in TD.

In an effort to map a bigger part of a country, an atlas[4] (a collection of maps) has proved to be helpful. A good atlas should be consistent at all overlaps: if some part of the land happens to be on two (or more) maps (close to the margins, as a rule), information obtained from them must not be mutually contradictory.

If we enlarge the region to be mapped (district $\mapsto$ country $\mapsto$ continent, etc.), we first observe annoying *metric* properties of the maps – the continents become (in comparison with their shape on the globe) somewhat deformed and the intuitive estimation of the *distances* becomes unreliable. This is a manifestation of the fact that ψ fails to be an *isometry* (see Section 4.6); as a matter of fact such an isometry (of a part of the sphere to a part of a sheet of paper) *does not exist* at all.[5] Topologically, however, everything is still all right – even if TD = all of America, ψ still remains a homeomorphism (the latter *need not* preserve distances). But even this ceases to be the case abruptly at the moment we try to display *all* the globe on a *single* map. It turns out, once again, that such a map (a bijective and continuous map of a sphere onto a plane) *does not exist*; that is to say, *more than* one single map – an atlas – is *inevitable*. An optimistic element in these contemplations lies in the fact that *in spite of* the topological complexity of the sphere S^2 (as compared with the plane), its mapping is fairly simple when an atlas containing *several* maps is used. In a similar way one can construct (highly practical) atlases of some other two-dimensional surfaces, like T^2 = the surface of a tire (repairmen in a tire service will then be happy to mark the exact position of a puncture into this atlas) or the exotic (1.5.9) Klein bottle K^2 (appreciated by orienteering fans, mainly in sci-fi).

[4] Atlas, the brother of Prometheus, hero of Greek mythology, keeps (as he used to do) the cope of heaven on his shoulders on the title page of a series of detailed maps of various parts of Europe. They were published in 1595, one year after the death of the author, Gerhard Kramer (Gerardus Mercator in Latin). Since then, *every* series of maps has been called an "atlas."

[5] There are several characteristics preserved by isometries and the sphere and the sheet of paper *differ* in some of them (see, e.g., the result of the computation of the Lie algebras of Killing fields in (4.6.10) and (4.6.13) or of the scalar curvature in (15.6.11)).

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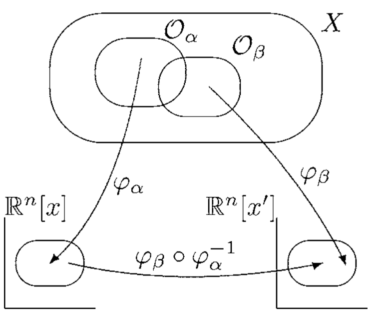
The aim, now, will be to formalize the idea of an atlas. This will result in the definition of the crucial concept of a smooth manifold.

Let $(X, \{\tau\})$ be a topological space and $\mathcal{O} \subset X$ an open set. A homeomorphism



$$\varphi : \mathcal{O} \to \mathbb{R}^n[x^1, \dots, x^n]$$

is called a *chart*, or alternatively *local coordinates*. Each point $x \in \mathcal{O} \subset X$ is then uniquely associated with an n-tuple of real numbers – its coordinates. The set $\mathcal{O}$ is known as a *coordinate patch* in this context. So far we have introduced coordinates in a single coordinate patch – in $\mathcal{O}$. If we want to assign coordinates to all points from X, we need an *open covering* $\{\mathcal{O}_\alpha\}$ of the space X (i.e. $\bigcup_\alpha \mathcal{O}_\alpha = X$) and local coordinates for each domain $\mathcal{O}_\alpha$

$$\varphi_\alpha : \mathcal{O}_\alpha \to \mathbb{R}^n$$

(n being the same for all α). A collection of charts $\mathcal{A} \equiv \{\mathcal{O}_\alpha, \varphi_\alpha\}$ is called an *atlas* on X. If the intersection $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ is non-empty, a map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : A \to \mathbb{R}^n, \qquad A \equiv \varphi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathbb{R}^n$$

called a *change of coordinates* is induced. Since it is a map of Cartesian spaces (see Section 1.2), it makes sense to talk about its class of smoothness. Automatically (check (1.1.3)) its class is C^0, but it might be higher. If, given an atlas, *all* maps of this type happen to be C^k or higher, it is called a *C^k-atlas* $\mathcal{A}$.

An atlas may be supplemented by additional maps, provided that the consistency with the maps already present is assured. A map

$$\mu : \mathcal{O} \to \mathbb{R}^n$$

is said to be *C^k-related* (and it may be added to $\mathcal{A}$), if it is consistent with all maps $(\mathcal{O}_\alpha, \varphi_\alpha)$ on the intersections $\mathcal{O} \cap \mathcal{O}_\alpha$, i.e. if the class of the map $\varphi_\alpha \circ \mu^{-1}$ is C^k or higher. If a C^k-atlas $\mathcal{A}$ is supplemented consecutively with *all* maps, we are left with a unique *maximal C^k-atlas* $\hat{\mathcal{A}}$. This in turn endows X with a *C^k-structure*. A pair $(X, \hat{\mathcal{A}})$ is called an (n-dimensional) *C^k-manifold* (in particular, topological, differentiable, ..., smooth, analytic). In this book we will be concerned exclusively with[6] *smooth manifolds*, or here and there (when Taylor series are used) even *analytic manifolds*. The essential structure to be used implicitly throughout the book and assumed to be available in all discussions and constructions is the *smooth structure* on a manifold X.

Since an atlas $\mathcal{A}$ leads to the unique maximal atlas $\hat{\mathcal{A}}$, for the practical construction of a manifold it suffices to give the atlas $\mathcal{A}$. In spite of this fact the definition of a manifold

[6] This highly convenient option is offered by the result of the Whitney ("embedding") theorem, to be mentioned later, see Section 1.4.

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refers to the maximal atlas. This emphasizes the formal *equality of rights* of all charts (local coordinates). The constitution (= definition) unambiguously states that the initial charts from $\mathcal{A}$ are by no means privileged in $\hat{\mathcal{A}}$ with respect to those coming later (so that there is no fear of them usurping a privileged position at any later moment). This does not at all mean that privileged coordinates are of no importance in differential geometry. If the *smooth* structure is the *only* structure available, all charts are to be treated equally. In applications, on the other hand, there are typically additional structures on manifolds. Then, of course, particular coordinates tailored to these structures (*adapted coordinates*) would play a privileged role from the *practical* point of view.

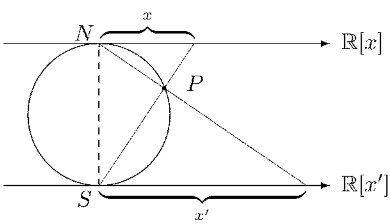
The simplest n-dimensional manifold is clearly $\mathbb{R}^n$ itself. A possible atlas is comprised of a *single* chart, given by the *identity* map

$$\varphi \equiv \mathrm{id} : \mathbb{R}^n[x^1, \dots, x^n] \to \mathbb{R}^n[x^1, \dots, x^n]$$

This atlas is trivially smooth (or analytic as well; there are no intersections to spoil it) and the maximal atlas generated by this atlas defines the *standard smooth structure* in $\mathbb{R}^n$. Any other chart from this atlas corresponds to *curvilinear coordinates* in $\mathbb{R}^n$ (like the polar coordinates in a part of the plane $\mathbb{R}^2$).

The next two exercises deal with the construction of smooth atlases on spheres and projective spaces.

$\boxed{1.3.1}$ On a circle S^1 of radius R we introduce local coordinates x, x' as shown on the figure (this is called the *stereographic projection*). On higher-dimensional spheres $S^2, \dots, S^n$ a natural generalization of this idea results in coordinates $\mathbf{r}, \mathbf{r}'$. Verify that:



(i) on the intersection of the patches, where the primed and unprimed coordinates are in operation, we find for S^1 and S^n respectively the following explicit transition relations:

$$x' = \frac{(2R)^2}{x} \qquad \mathbf{r}' = \frac{(2R)^2}{r}\frac{\mathbf{r}}{r}$$

(ii) in this way an analytic atlas composed of two charts has been constructed on S^n – the sphere S^n is thus an n-dimensional analytic manifold;

(iii) if the complex coordinates z and z' are introduced on S^2

$$\mathbf{r} \leftrightarrow (x, y) \leftrightarrow z \equiv x + iy \qquad \mathbf{r}' \leftrightarrow (x', y') \leftrightarrow z' \equiv x' + iy'$$

then the transition relations are

$$z' = (2R)^2/\bar{z} \qquad \bar{z} \equiv x - iy$$

Hint: on S^n a projection is to be performed onto n-dimensional mutually parallel *planes*, touching the north and south poles respectively (in these planes $\mathbf{r} \equiv (x^1, \dots, x^n)$ represent common Cartesian coordinates centered at the poles). Then $\mathbf{r}' = \lambda\mathbf{r}$ and one easily finds λ

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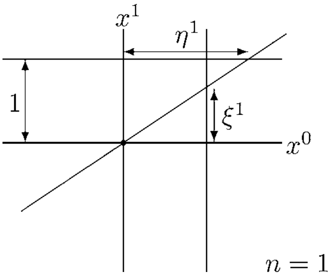
from the observation that in the (two-dimensional) plane given by the poles and the point P the situation reduces to S^1. □

$\boxed{1.3.2}$ The real *projective space* $\mathbb{R}P^n$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. The complex projective space $\mathbb{C}P^n$ is introduced similarly – one should replace $\mathbb{R} \mapsto \mathbb{C}$ in the preceding definition. (Here, a complex line consists of all *complex* multiples of a fixed (non-vanishing) complex vector (point of $\mathbb{C}^{n+1}$) z, so that it is a *two-dimensional* object from a real point of view.)

(i) Introduce the structure of an n-dimensional smooth manifold (= local coordinates) on $\mathbb{R}P^n$.
(ii) The same for $\mathbb{C}P^n$ (it is $2n$-dimensional).
(iii) Show that the states of an n-level system in quantum mechanics are in one-to-one correspondence with the points of $\mathbb{C}P^{n-1}$.
(iv) Show that $\mathbb{C}P^1 = S^2$ (in the sense of (1.4.7)) $\Rightarrow$ states with spin $\frac{1}{2}$ correspond to unit vectors $\mathbf{n}$ in $\mathbb{R}^3$.



Hint: (i) one line (a point from $\mathbb{R}P^n$) consists of those points of $\mathbb{R}^{n+1}$ which may be obtained from a fixed $(x^0, x^1, \dots, x^n)$ using the freedom $(x^0, x^1, \dots, x^n) \sim (\lambda x^0, \dots, \lambda x^n)$; in the part of $\mathbb{R}^{n+1}$ where $x^0 \neq 0$ the freedom enables one to make 1 from the first entry of the array (visually this means that the point of intersection of the line with the plane $x^0 = 1$ has been used as a representative of the line); the other n numbers are to be used as local coordinates on $\mathbb{R}P^n$ (they are the coordinates in the plane $x^0 = 1$ mentioned above; see the figure for $n = 1$, try to draw the case $n = 2$): $(x^0, x^1, \dots, x^n) \sim (\lambda x^0, \dots, \lambda x^n) \sim (1, \xi^1, \dots, \xi^n)$ for $x^0 \neq 0$, $\Rightarrow (\xi^1, \dots, \xi^n)$ are coordinates (there); in this way obtain step-by-step $(n+1)$ charts,[7] with the last one coming from $(x^0, x^1, \dots, x^n) \sim (\lambda x^0, \dots, \lambda x^n) \sim (\eta^1, \dots, \eta^n, 1)$ for $x^n \neq 0$; (ii) in full analogy, $\xi, \dots, \eta$ are now *complex* n-tuples, giving rise to $2n$ real coordinates; (iii) two non-vanishing vectors in a Hilbert space, one of them being a complex constant multiple of the other, correspond to a single state; (iv) spin $\frac{1}{2}$ is a two-level system. □

• From two given manifolds $(X, \hat{\mathcal{A}})$ and $(Y, \hat{\mathcal{B}})$, we can form a new manifold called the *Cartesian product*. This new manifold is denoted by the symbol $X \times Y$. As a set, it is the Cartesian product $X \times Y$ (points being ordered pairs), an atlas is constructed in the exercise.

$\boxed{1.3.3}$ Let $(X, \hat{\mathcal{A}})$ and $(Y, \hat{\mathcal{B}})$ be smooth manifolds and let

$$\varphi_\alpha : \mathcal{O}_\alpha \to \mathbb{R}^n \qquad \psi_a : \mathcal{S}_a \to \mathbb{R}^m$$

represent two charts on X and Y respectively. Show that

$$\begin{aligned} \varphi_\alpha \times \psi_a : \mathcal{O}_\alpha \times \mathcal{S}_a &\to \mathbb{R}^{m+n} \\ (x, y) &\mapsto (\varphi_\alpha(x), \psi_a(y)) \in \mathbb{R}^{n+m} \qquad x \in \mathcal{O}_\alpha, \quad y \in \mathcal{S}_a \end{aligned}$$

[7] In this context the coordinates $(x^0, x^1, \dots, x^n)$ in $\mathbb{R}^{n+1}$ are said to be the *homogeneous coordinates* (of the points in $\mathbb{R}P^n$). Note that they are *not* local coordinates on $\mathbb{R}P^n$ in the sense of the definition of a manifold, since they are not in *one-to-one* correspondence with the points (they *are* official coordinates only in $\mathbb{R}^{n+1}$).

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