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139 Typical Dynamics of Volume Preserving Homeomorphisms

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Dedicated to the memory of John Oxtoby and Stan Ulam

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Historical Preface

This monograph covers the authors' work over the past twenty five years on generalizing the classical results of John Oxtoby and Stan Ulam on the typical dynamical behavior of manifold homeomorphisms which preserve a fixed measure. In the main text of the book we will take a logical rather than historical perspective, designed to give the reader a concise and unified treatment of results we obtained in a series of articles that were written before the overall structure of the theory was clear. However, since the true significance of this field of study can be understood only from a historical perspective, we devote this preface to a discussion of the problem considered by Oxtoby and Ulam when they were Junior Fellows at Harvard in the 1930s, and of their accomplishment in its solution. We shall use their own words where possible.

The origins of Ergodic Theory lie in the study of physical systems which evolve in time as solutions to certain differential equations. Such systems can be initially described by parameters giving the states of the system as points in Euclidean n-space. Taking conservation laws into account, the phase space may be decomposed into lower dimensional manifolds. Regularities in the differential equations obeyed by the system are reflected in the differentiability or the continuity of the flow that describes the evolution of the system over time. Furthermore, Liouville's Theorem ensures that for Hamiltonian systems this flow has an invariant measure. Thus one is led in a natural way from the underlying physics to the study of measure preserving manifold homeomorphisms or diffeomorphisms. As the latter have received much attention we will confine our attention here to the case of homeomorphisms.

An important historical assumption that was often made in the study of such systems was the so called 'ergodic hypothesis' of statistical mechanics, as described by Oxtoby and Ulam in their 1941 paper [88, p. 874]:

> In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time-averages may be replaced by space-averages. To justify this interchange, a number of hypotheses were proposed, variously known as ergodic or quasi-ergodic hypotheses. . . . A rigorous discussion of the precise conditions under which the interchange was admissible was only made possible in 1931 by the ergodic theorem of Birkhoff. This established the *existence* of the time-averages in question . . . and showed that . . . the interchange is permissible if and only if the flow in phase space is *metrically transitive* [the older term for *ergodic*]. A transformation or a flow is metrically transitive [ergodic] if there do not exist two disjoint invariant sets both having positive measure. Thus the effect of the ergodic theorem was to replace the ergodic hypothesis by the hypothesis of metrical transitivity [ergodicity].

An important question in the 1930s was consequently the determination of which known transformations were ergodic, and more generally, which manifolds could support an ergodic homeomorphism. Aside from the pure existence question, both Birkhoff and Hopf had conjectured that ergodicity was the *general case* for transformations, in some unspecified sense. A natural setting at that time in which to make their conjecture precise was Baire's notion of category. In this topological context, ergodic homeomorphisms represent the general case if the nonergodic ones constitute a set of *first category* (that is, the union of countably many nowhere dense sets).

When Oxtoby and Ulam were Junior Fellows at Harvard in the late 1930s, the main problem they worked on was the determination of those (connected) compact manifolds for which ergodicity was the general case for measure preserving homeomorphisms. Their main finding was that ergodicity is the general case for *all* compact manifolds, or as they put it, 'the hypothesis of metrical transitivity in dynamics involves no *topological* contradiction'. John Oxtoby told us that during this period G. D. Birkhoff was their main source of problems (in particular this one) and Marshall Stone was the main source of techniques regarding their solution.

Ulam describes his work with Oxtoby on this problem in his autobiography *Adventures of a Mathematician* [103], in the chapter Harvard Years, 1936–1939:

> In order to complete the foundation of the ideas of statistical mechanics connected with the ergodic theorem, it was necessary to prove the existence, and

what is more, the prevalence of ergodic transformations. G. D. Birkhoff himself had worked on special cases in dynamical problems, but there were no general results. We wanted to show that on every manifold (a space representing the possible states of a dynamical system) – the kind used in statistical mechanics – such ergodic behavior is the rule. ... We discussed various approaches to a possible construction of these transformations. ... We kept G. D. Birkhoff informed of the status of our attacks on the problem. ... He would check what I told him with Oxtoby, a more cautious person. It took us more than two years to break through and to finish a long paper [88] which appeared in *The Annals of Mathematics* in 1941, and which I consider one of the more important results that I had a part in.

The result of Oxtoby and Ulam that ergodicity is generic for measure preserving homeomorphisms of compact manifolds has been generalized in two ways. The first direction in which their result extends is that the property of ergodicity has been generalized to more specialized measure theoretic behavior. This was first done by Katok and Stepin [76], who in 1970 proved that weak mixing homeomorphisms are also generic. To put Katok and Stepin's result in a historical context, we note that subsequent to Oxtoby and Ulam's 1941 paper, Paul Halmos published two papers: the first [69] in 1944 showed that ergodicity is generic in the weak topology in the space of all measure preserving bijections (called automorphisms) of a measure space; in a second paper that year [70], Halmos proved that weak mixing is also a generic property for measure preserving bijections. In describing the relation between his theorem [69, Theorem 6] on ergodicity being generic for measure preserving bijections and Oxtoby and Ulam's theorem on generic ergodicity for measure preserving homeomorphisms, Halmos notes [69, p. 2, footnote 1]:

The first theorem of this type is due to J. C. Oxtoby and S. M. Ulam ... Their topology is however, very different from mine and depends on the topological and metric (as opposed to purely measure theoretic) structure of the underlying space.

Further on in his paper, Halmos states [69, p. 12]:

... there is, however, no implication between [Halmos's] Theorem 6 and the corresponding result of Oxtoby and Ulam: they define a stronger topology and I consider a wider class of transformations.

Halmos's statement notwithstanding, the first author (S. Alpern) showed that in fact any measure theoretic property which is generic for abstract measure preserving automorphisms is also generic for measure preserving homeomorphisms of compact manifolds. Thus Alpern's result related the two 1944 papers of Halmos in the former context (proofs that ergodicity and then weak mixing were generic) to the work of Oxtoby–Ulam and Katok–Stepin. This generalization of the Oxtoby–Ulam Theorem to all typical measure theoretic properties is covered in the first half of the book (Parts I and II), which is devoted to compact manifolds. In fact most of the theory is developed in Part I in the special context of volume preserving homeomorphisms of the unit n-cube. Part II shows how these results may be generalized to homeomorphisms of a compact manifold which preserve a certain finite measure. Some of the more elementary aspects of this work can be very simply developed using the ideas of Lax [80] on discrete approximation of measure preserving homeomorphisms, including some applications to fixed point theory. However, the main logical development is independent of these combinatorial notions and uses instead the idea of viewing the space of measure preserving *homeomorphisms* of a manifold as being embedded in the larger space consisting of all *bijections* of the manifold which preserve that measure. Properties of this embedding are established through a Lusin Theorem for measure preserving homeomorphisms.

The second direction of generalization of the result of Oxtoby and Ulam, covered in Part III, is the removal of the compactness assumption on the underlying manifold, and the concomitant consideration of infinite preserved measures. Although Besicovitch had established the existence of a transitive homeomorphism of the plane in 1937, the corresponding result for ergodicity was not established until 1979, when Prasad [96] showed that in fact ergodicity is generic for volume preserving homeomorphisms of R^n. However, it soon became clear that unlike the compact case, in which all manifolds supported generic ergodicity, not all noncompact manifolds had this property. The search for the relevant manifold property which determined the supported dynamical behavior then centered on the so called *ends* of the manifold, roughly speaking, the distinct ways of going to infinity. The purely measure theoretic underpinning for the infinite measure work was established by Choksi and Kakutani [50], who showed in 1979 that ergodicity is a typical property for measure preserving bijections of an *infinite* Lebesgue space.

For noncompact manifolds, the space of measure preserving homeomorphisms divides into components according to the induced homeomorphism of the set of ends. We find, for example, that if the induced end homeomorphism is transitive then ergodicity is generic within such a component. Furthermore, if the induced end homeomorphism is topologically weak mixing, then any property generic for measure preserving transformations of an infinite Lebesgue space is generic within the component. A fuller description of the authors' work on noncompact manifolds is contained in the Introduction to Part III.

This book covers only those aspects of the field of measure preserving homeomorphisms of a manifold that involve *typical* properties of such transformations. So for example we do not discuss the important result of Lind and Thouvenot [83] on ergodic theoretic behavior represented by some measure preserving torus homeomorphisms, because the behavior they demonstrate is not typical.

Our aim is to give a streamlined approach to our work in this area, from a perspective only recently reached and not fully appreciated in our articles on the subject. As this is a work centered on the interaction of measure and topology, we have given full proofs of all results that combine these two fields (the core of the theory) while leaving out some proofs of results that fall fully within measure theory or manifold topology.

Most of the work described in the first two parts of the book was carried out under the guidance and encouragement of John Oxtoby. The early work of Alpern in this area also benefited from discussions with Stan Ulam. Aside from these two founders of the field of measure preserving homeomorphisms, the four mathematicians whose ideas most influenced this work are Jal Choksi, Robert D. Edwards, Shizuo Kakutani, and Peter Lax.

General Outline

The book as a whole gives a unified presentation of the authors' work on establishing conditions under which an ergodic theoretic dynamical property is typical in the space $\mathcal{M}[X, \mu]$ consisting of all homeomorphisms of a sigma compact manifold X which preserve a fixed nonatomic Borel measure μ. The first half of the book, comprising Parts I and II, covers the first author's work on compact manifolds (for which μ is necessarily finite). For clarity of exposition the material in the first eight chapters (Part I) is presented for the special case where the compact manifold X is simply the unit n-dimensional cube I^n and the measure μ is n-dimensional Lebesgue measure (volume). In Part II, comprising Chapters 9 and 10, we show how the results obtained for the cube hold as well for arbitrary compact manifolds. The second half of the book, Part III (Chapters 11–17), describes the work of both authors in extending the earlier work to the case where the manifold X is not compact (and μ may be infinite). In some cases the earlier work for the compact case cannot be extended, and we establish such negative results as well. In this half of the book the results depend in a significant way on the structure of the 'ends' of the manifold X, which are roughly the ways of going to infinity on the manifold. In particular, the ergodic theoretic properties of a μ-preserving homeomorphism h of a noncompact manifold X will depend on its induced action on the ends of X and on the net measure that it flows into each end. Following Part III, there are two appendices. Appendix 1 is mainly concerned with presenting a purely measure theoretic result of the first author, which we call the Multiple Tower Rokhlin Theorem, as it generalizes a similar result due to Rokhlin and Halmos for a single tower. Corollaries of this theorem, as well as that of an infinite measure version due to the authors and J. Choksi, are used extensively in the main part of the book. Appendix 2 is the only chapter of the book which is not based on the work of the authors. It presents theorems, due to von Neumann and Oxtoby–Ulam (for compact manifolds), to Oxtoby–Prasad (for the Hilbert cube), and to Berlanga and Epstein (for sigma compact manifolds), which give necessary and sufficient conditions for two measures μ and ν on a manifold to be 'homeomorphic'. This means that for some self-homeomorphism of the manifold we have $\mu(A) = \nu(h(A))$ for all Borel sets A.

Recalling the first sentence of this outline, we now say what we mean by an 'ergodic theoretic dynamical property' and by 'typical'. There are of course many types of properties that a measure preserving homeomorphism might possess. For example if the manifold is the cube, it must have a fixed point. However, this property 'lives' on a set of measure zero, and we are concerned mainly with properties that 'live' on a set of full measure, such as ergodicity. More precisely, we are concerned with properties that can be defined in the larger space $\mathcal{G}[X, \mu]$ consisting of all μ-preserving bijections of the manifold X viewed simply as a measure space, where the manifold structure is irrelevant. Examples of such measure theoretic properties are ergodicity, weak mixing, and zero entropy. In order to say what we mean by a 'typical' property, we must endow the space $\mathcal{M}[X, \mu]$ with a topology, which we take to be the uniform topology when X is compact, and more generally the topology of uniform convergence on compact sets when it is not. Then we say a property is typical in $\mathcal{M}[X, \mu]$ if the homeomorphisms possessing it contain a dense G_δ subset. In a similar fashion we will say that a measure theoretic property is typical in the space $\mathcal{G}[X, \mu]$ if it contains a dense G_δ subset of that space, with respect to a commonly used topology called the weak topology.

The main aim of the first half of the book, carried out in Parts I and II, is the derivation of the first author's result that any measure theoretic property (such as ergodicity or weak mixing) which is typical in the measure theoretic context (that is, in $\mathcal{G}[X, \mu]$) is also typical for measure preserving homeomorphisms (that is, in $\mathcal{M}[X, \mu]$ for compact manifolds X). The aims of the second half are more varied. We develop the results of both authors in establishing positive and negative results regarding the typicality of certain properties on various manifolds. Unlike the fairly universal result stated above for compact manifolds (universal in that all properties and all manifolds are treated similarly), we find that our results for noncompact manifolds depend both on the property and on the manifold. For Euclidean space R^n, the second author showed that ergodicity is typical for volume preserving homeomorphisms. This is the first result presented in Part III. We then present examples of manifolds where ergodicity is not typical. After an extensive treatment of the interaction of ends and measures, we obtain a synthesis of the positive and negative results regarding ergodicity: A homeomorphism h in $\mathcal{M}[X, \mu]$ is the limit of ergodic homeomorphisms if and only if it does not compress any set of ends of X (into a proper subset of itself) and it does not induce a positive flow of measure into any set of ends. As the identity homeomorphism on X has these properties, it follows that any manifold supports an ergodic homeomorphism. We then consider more general properties, and show that any property typical in $\mathcal{G}[X, \mu]$ is typical in a certain nonempty closed subspace of $\mathcal{M}[X, \mu]$, and is consequently possessed by some μ-preserving homeomorphism of X. In particular there are weak mixing homeomorphisms of any sigma compact manifold (X, μ).

Despite our earlier disclaimer regarding properties that live on sets of measure zero, the book does include results for such a property, namely *maximal chaos*. This is a topological property introduced by the authors which entails topological transitivity, dense periodic points, and a maximal form of sensitive dependence on initial conditions. As such, it is a strictly stronger property than Devaney's version of chaos. In Chapter 4 we establish that homeomorphisms with maximal chaos are dense in $\mathcal{M}[X, \mu]$ when X is compact, and in Chapter 17 we establish that for arbitrary sigma compact manifolds such homeomorphisms are dense in a nonempty subset of $\mathcal{M}[X, \mu]$. In particular, any sigma compact manifold supports a maximally chaotic homeomorphism. In addition to the topological property of chaos, we also apply our techniques to the fixed point property. In Chapter 5 we look at the relationship between this property and area preservation, for various 2-dimensional manifolds. We apply an approximation technique due to Peter Lax to give simple proofs of both the Poincaré–Birkhoff Theorem and the Conley–Zehnder–Franks Theorem, results which assert the existence of fixed points for area preserving homeomorphisms of the annulus and torus, respectively, under some additional hypotheses. The same technique (involving the Marriage Theorem) is also used in Chapter 7 to give a new proof that ergodicity is typical for volume preserving homeomorphisms of the cube.

The material above is meant to give the reader a very informal idea of the main results covered in this book. For a slightly more detailed presentation of the main results, the reader is referred to Section 1.3

(for results on compact manifolds) and Section 11.3 (for noncompact manifolds).

We wish to assure readers who come to this book with little or no familiarity with the fields of ergodic theory or measure theory that no prior knowledge of these fields is required. All the ergodic theoretic notions that we will use will be explained and defined when they are needed.