CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

EDITORIAL BOARD
B. BOLLOBAS, W. FULTON, A. KATOK, F. KIRWAN, P. SARNAK

Lectures in Logic and Set Theory Volume 2

This two-volume work bridges the gap between introductory expositions of logic or set theory on one hand, and the research literature on the other. It can be used as a text in an advanced undergraduate or beginning graduate course in mathematics, computer science, or philosophy. The volumes are written in a user-friendly conversational lecture style that makes them equally effective for self-study or class use.

Volume 2, on formal (ZFC) set theory, incorporates a self-contained “Chapter 0” on proof techniques so that it is based on formal logic, in the style of Bourbaki. The emphasis on basic techniques will provide the reader with a solid foundation in set theory and sets a context for the presentation of advanced topics such as absoluteness, relative consistency results, two expositions of Gödel’s constructible universe, numerous ways of viewing recursion, and a chapter on Cohen forcing.

George Tourlakis is Professor of Computer Science at York University of Ontario.
LECTURES IN LOGIC
AND SET THEORY

Volume 2: Set Theory

GEORGE TOURLAKIS
York University
To the memory of my parents
Contents

Preface  xi
I  A Bit of Logic: A User's Toolbox  1
I.1  First Order Languages  7
I.2  A Digression into the Metatheory: Informal Induction and Recursion  20
I.3  Axioms and Rules of Inference  29
I.4  Basic Metatheorems  43
I.5  Semantics  53
I.6  Defined Symbols  66
I.7  Formalizing Interpretations  77
I.8  The Incompleteness Theorems  87
I.9  Exercises  94
II  The Set-Theoretic Universe, Na"ively  99
II.1  The “Real Sets”  99
II.2  A Na"ive Look at Russell’s Paradox  105
II.3  The Language of Axiomatic Set Theory  106
II.4  On Names  110
III  The Axioms of Set Theory  114
III.1  Extensionality  114
III.2  Set Terms; Comprehension; Separation  119
III.3  The Set of All Urelements; the Empty Set  130
III.4  Class Terms and Classes  134
III.5  Axiom of Pairing  145
III.6  Axiom of Union  149
III.7  Axiom of Foundation  156
III.8  Axiom of Collection  160
III.9  Axiom of Power Set  178
Contents

III.10 Pairing Functions and Products 182
III.11 Relations and Functions 193
III.12 Exercises 210

IV The Axiom of Choice 215
IV.1 Introduction 215
IV.2 More Justification for AC; the “Constructible” Universe Viewpoint 218
IV.3 Exercises 229

V The Natural Numbers; Transitive Closure 232
V.1 The Natural Numbers 232
V.2 Algebra of Relations; Transitive Closure 253
V.3 Algebra of Functions 272
V.4 Equivalence Relations 276
V.5 Exercises 281

VI Order 284
VI.1 PO Classes, LO Classes, and WO Classes 284
VI.2 Induction and Inductive Definitions 293
VI.3 Comparing Orders 316
VI.4 Ordinals 323
VI.5 The Transfinite Sequence of Ordinals 340
VI.6 The von Neumann Universe 358
VI.7 A Pairing Function on the Ordinals 373
VI.8 Absoluteness 377
VI.9 The Constructible Universe 395
VI.10 Arithmetic on the Ordinals 410
VI.11 Exercises 426

VII Cardinality 430
VII.1 Finite vs. Infinite 431
VII.2 Enumerable Sets 442
VII.3 Diagonalization; Uncountable Sets 451
VII.4 Cardinals 457
VII.5 Arithmetic on Cardinals 470
VII.6 Cofinality; More Cardinal Arithmetic; Inaccessible Cardinals 478
VII.7 Inductively Defined Sets Revisited; Relative Consistency of GCH 494
VII.8 Exercises 512

VIII Forcing 518
VIII.1 PO Sets, Filters, and Generic Sets 520
VIII.2 Constructing Generic Extensions 524
Contents

VIII.3 Weak Forcing 528
VIII.4 Strong Forcing 532
VIII.5 Strong vs. Weak Forcing 543
VIII.6 $M[G]$ Is a CTM of ZFC If $M$ Is 544
VIII.7 Applications 549
VIII.8 Exercises 558
Bibliography 560
List of Symbols 563
Index 567
Preface

This volume contains the basics of Zermelo-Fraenkel axiomatic set theory. It is situated between two opposite poles: On one hand there are elementary texts that familiarize the reader with the vocabulary of set theory and build set-theoretic tools for use in courses in analysis, topology, or algebra – but do not get into metamathematical issues. On the other hand are those texts that explore issues of current research interest, developing and applying tools (constructibility, absoluteness, forcing, etc.) that are aimed to analyze the inability of the axioms to settle certain set-theoretic questions.

Much of this volume just “does set theory”, thoroughly developing the theory of ordinals and cardinals along with their arithmetic, incorporating a careful discussion of diagonalization and a thorough exposition of induction and inductive (recursive) definitions. Thus it serves well those who simply want tools to apply to other branches of mathematics or mathematical sciences in general (e.g., theoretical computer science), but also want to find out about some of the subtler results of modern set theory.

Moreover, a fair amount is included towards preparing the advanced reader to read the research literature. For example, we pay two visits to Gödel’s constructible universe, the second of which concludes with a proof of the relative consistency of the axiom of choice and of the generalized continuum hypothesis with ZF. As such a program requires, I also include a thorough discussion of formal interpretations and absoluteness. The lectures conclude with a short but detailed study of Cohen forcing and a proof of the non-provability in ZF of the continuum hypothesis.

The level of exposition is designed to fit a spectrum of mathematical sophistication, from third-year undergraduate to junior graduate level (each group will find here its favourite chapters or sections that serve its interests and level of preparation).
The volume is self-contained. Whatever tools one needs from mathematical logic have been included in Chapter I. Thus, a reader equipped with a combination of sufficient mathematical maturity and patience should be able to read it and understand it. There is a trade-off: the less the maturity at hand, the more the supply of patience must be. To pinpoint this “maturity”: At least two courses from among calculus, linear algebra, and discrete mathematics at the junior level should have exposed the reader to sufficient diversity of mathematical issues and proof culture to enable him or her to proceed with reasonable ease.

**A word on approach.** I use the Zermelo-Fraenkel axiom system with the axiom of choice (AC). This is the system known as ZFC. As many other authors do, I simplify nomenclature by allowing “proper classes” in our discussions as part of our metalanguage, but not in the formal language.

I said earlier that this volume contains the “basics”. I mean this characterisation in two ways: One, that all the fundamental tools of set theory as needed elsewhere in the mathematical sciences are included in detailed exposition. Two, that I do not present any applications of set theory to other parts of mathematics, because space considerations, along with a decision to include certain advanced relative consistency results, have prohibited this.

“Basics” also entails that I do not attempt to bring the reader up to speed with respect to current research issues. However, a reader who has mastered the advanced metamathematical tools contained here will be able to read the literature on such issues.

The title of the book reflects two things: One, that all good short titles are taken. Two, more importantly, it advertises my conscious effort to present the material in a conversational, user-friendly lecture style. I deliberately employ classroom mannerisms (such as “pauses” and parenthetical “why”s, “what if”s, and attention-grabbing devices for passages that I feel are important). This aims at creating a friendly atmosphere for the reader, especially one who has decided to study the topic without the guidance of an instructor. Friendliness also means steering clear of the terse axiom-definition-theorem recipe, and explaining how some concepts were arrived at in their present form. In other words, what makes things tick. Thus, I approach the development of the key concepts of ordinals and cardinals, initially and tentatively, in the manner they were originally introduced by Georg Cantor (paradox-laden and all). Not only does this afford the reader an understanding of why the modern (von Neumann) approach is superior (and contradiction-free), but it also shows what it tries to accomplish. In the same vein, Russell’s paradox is visited no less than three
Preface

times, leaving us in the end with a firm understanding that it has nothing to do with the “truth” or otherwise of the much-maligned statement “$x \in x$” but it is just the result of a diagonalization of the type Cantor originally taught us.

A word on coverage. Chapter I is our “Chapter 0”. It contains the tools needed to enable us do our job properly – a bit of mathematical logic, certainly no more than necessary. Chapter II informally outlines what we are about to describe axiomatically: the universe of all the “real” sets and other “objects” of our intuition, a caricature of the von Neumann “universe”. It is explained that the whole fuss about axiomatic set theory† is to have a formal theory derive true statements about the von Neumann sets, thus enabling us to get to know the nature and structure of this universe. If this is to succeed, the chosen axioms must be seen to be “true” in the universe we are describing.

To this end I ensure via informal discussions that every axiom that is introduced is seen to “follow” from the principle of the formation of sets by stages, or from some similarly plausible principle devised to keep paradoxes away. In this manner the reader is constantly made aware that we are building a meaningful set theory that has relevance to mathematical intuition and expectations (the “real” mathematics), and is not just an artificial choice of a contradiction-free set of axioms followed by the mechanical derivation of a few theorems.

With this in mind, I even make a case for the plausibility of the axiom of choice, based on a popularization of Gödel’s constructible universe argument. This occurs in Chapter IV and is informal.

The set theory we do allows atoms (or Urelemente),‡ just like Zermelo’s. The re-emergence of atoms has been defended aptly by Jon Barwise (1975) and others on technical merit, especially when one does “restricted set theories” (e.g., theory of admissible sets).

Our own motivation is not technical; rather it is philosophical and pedagogical. We find it extremely counterintuitive, especially when addressing undergraduate audiences, to tell them that all their familiar mathematical objects – the “stuff of mathematics” in Barwise’s words – are just perverse “box-in-a-box-in-a-box . . .” formations built from an infinite supply of empty boxes. For example, should I be telling my undergraduate students that their familiar number “2” really is just a short name for something like “$igvee$

† O.K., maybe not the whole fuss. Axiomatics also allow us to meaningfully ask, and attempt to answer, metamathematical questions of derivability, consistency, relative consistency, independence. But in this volume much of the fuss is indeed about learning set theory.
‡ Allows, but does not insist that there are any.
Some mathematicians have said that set theory (without atoms) speaks only of *sets* and it chooses *not* to speak about objects such as cows or fish (colourful terms for urelements). Well, it does too! Such (“atomless”) set theory is known to be perfectly capable of *constructing* “artificial” cows and fish, and can then proceed to talk about such animals as much as it pleases.

While atomless ZFC has the ability to construct or codify all the familiar mathematical objects in it, it does this so well that it betrays the prime directive of the axiomatic method, which is to have a theory that *applies* to diverse concrete (*meta* – i.e., outside the theory and in the realm of “everyday math”) mathematical systems. Group theory and projective geometry, for example, fulfill the directive.

In atomless ZFC the opposite appears to be happening: One is asked to *embed* the known mathematics into the formal system.

We prefer a set theory that allows both artificial and real cows and fish, so that when we want to illustrate a point in an example utilizing, say, the everyday set of integers, \( \mathbb{Z} \), we can say things like “let the atoms (be interpreted to) include the members of \( \mathbb{Z} \ldots \).”

But how about technical convenience? Is it not hard to include atoms in a formal set theory? In fact, not at all!

**A word on exposition devices.** I freely use a pedagogical feature that, I believe, originated in Bourbaki’s books – that is, marking an important or difficult topic by placing a “winding road” sign in the margin next to it. I am using here the same symbol that Knuth employed in his TeXbook, namely, \( \text{\textcircled{}} \), marking with it the beginning and end of an important passage.

Topics that are advanced, or of the “read at your own risk” type, *can be omitted without loss of continuity*. They are delimited by a double sign, \( \text{\textcircled{}}} \).

Most chapters end with several exercises. I have stopped making attempts to sort exercises between “hard” and “just about right”, as such classifications are rather subjective. In the end, I’ll pass on to you the advice one of my professors at the University of Toronto used to offer: “Attempt all the problems. Those you can do, don’t do. Do the ones you cannot”.

**What to read.** Just as in the advice above, I suggest that you read everything that you do not already know if time is no object. In a class environment the coverage will depend on class length and level, and I defer to the preferences of the instructor. I suppose that a fourth-year undergraduate audience ought to see the informal construction of the constructible universe in Chapter IV, whereas a graduate audience would rather want to see the formal version in Chapter VI. The latter group will probably also want to be exposed to Cohen forcing.
Acknowledgments. I wish to thank all those who taught me, a group that is too large to enumerate, in which I must acknowledge the presence and influence of my parents, my students, and the writings of Shoenfield (in particular, 1967, 1978, 1971).

The staff at Cambridge University Press provided friendly and expert support, and I thank them. I am particularly grateful for the encouragement received from Lauren Cowles and Caitlin Doggart at the initial (submission and refereeing) and advanced stages (production) of the publication cycle respectively.

I also wish to record my appreciation to Zach Dorsey of TechBooks and his team. In both volumes they tamed my English and \LaTeX, fitting them to Cambridge specifications, and doing so with professionalism and flexibility.

This has been a long project that would have not been successful without the support and understanding – for my long leaves of absence in front of a computer screen – that only one’s family knows how to provide.

I finally wish to thank Donald Knuth and Leslie Lamport for their typesetting systems \TeX and \LaTeX that make technical writing fun (and also empower authors to load the pages with \mathcal{Q} and other signs).

George Tourlakis

*Toronto, March 2002*