Cambridge University Press
978-0-521-15177-1 — The Mechanics of the Circulation
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Excerpt
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1

Particles and continuous materials

The science of mechanics comprises the study of motion (or equilibrium) and the forces which cause it. The blood moves in the blood vessels, driven by the pumping action of the heart; the vessel walls, being elastic, also move; the blood and the walls exert forces on each other, which influence their respective motions. Thus, in order to study the mechanics of the circulation, we must first understand the basic principles of the mechanics of fluids (e.g. blood), and of elastic solids (e.g. vessel walls), and the nature of the forces exerted between two moving substances (e.g. blood and vessel walls) in contact.

As well as studying the relatively large-scale behaviour of blood and vessel walls as a whole, we can apply the laws of mechanics to motions right down to the molecular level. Thus, 'mechanics' is taken here to include all factors affecting the transport of material, including both diffusion and bulk motion.

The study of mechanics began in the time of the ancient Greeks, with the formulation of 'laws' governing the motion of isolated solid bodies. The Greeks believed that, for a body to be in motion, a force of some sort had to be acting upon it all the time; the physical nature of this force, exerted for example on an arrow in flight, was mysterious. The need for such a force was related to one of the paradoxes of the Greek philosopher Zeno: that the arrow occupies a given position during one instant, yet is simultaneously moving to occupy a different position at a subsequent instant.

These matters were not fully resolved until the seventeenth century when Isaac Newton formulated his three *laws of motion*, which form the basis of all the mechanics described in this book. The laws refer to the motion of individual particles, which are defined as objects with *mass* (so that, for example, the Earth exerts a gravitational pull on them), but which occupy single points (that is, they have no size). Of course, every real body, even one as small as an atom or an electron, has a finite size, but the laws of particle mechanics can be directly applied both to real bodies in isolation (like the arrow of Zeno's paradox, or the Earth in its motion round the Sun, or an individual red blood cell) and to extensive regions of continuous matter which can be deformed into different configurations. Examples of such deformable materials include all elastic solids, like steel, rubber and blood vessel walls, and all fluids, like water, treacle, blood plasma and air. Both liquids and gases are described here as fluids, since the laws of motion are applied in exactly the same way to each.

Newton's laws can be applied to bodies of finite size because it can be proved that a body will move as if all of its mass and all the external forces acting on it were concentrated at one point. This point is called the *centre of mass*.[1] Thus, the flight of the centre of mass of Zeno's arrow is the same as that of a particle of the same mass, acted on by the same forces of gravity and air resistance. Similarly, the motion through space of the Moon, or the Earth, or another planet, can be described by particle mechanics. So can the motion of the centre of mass of a blood cell, as long as the forces exerted on it by the surrounding plasma are known. However, the tumbling of a red cell, or the rotation of the Earth about its axis, or any other motion of a body relative to its centre of mass, depends on the detailed shape of the body and cannot be described as if the body were a particle.

The application of Newton's laws to the motion of continuous deformable materials is more difficult to justify. It is bound up with the implicit assumption that the fluids and solids we are interested in are continuous materials. In fact, physicists have long known that all matter is made up of molecules, bound together in various configurations by forces of various strengths,[2] and consisting of numbers of atoms. These in turn consist of central nuclei, surrounded by clouds of electrons, moving in orbits whose diameters are large compared with those of the nuclei. The motion of electrons round a nucleus is analogous to that of the planets round the sun, and like the solar system, most of an atom (and hence most matter) consists of empty space. Some typical dimensions are given in **Table 1.1.** It might be supposed that each nucleus, and each electron, or each atom, or even each molecule could be regarded as a particle, and its motion under the influence of the intermolecular forces deduced from Newton's laws. However, in air at normal temperature and pressure, for example, there are roughly 10^{20} molecules per cubic centimetre, and the position of each one would have to be specified precisely. Such a task is virtually impossible. The fact that the spacing between molecules is usually very small compared with the dimensions of the natural or experimental regions of fluid whose motion we wish to describe (see **Table 1.1**) indicates how we can overcome the difficulty. We may suppose the material to be divided up into a large number of elements whose dimensions are very small

[1] The centre of mass of a body is the same as its centre of gravity: if in the region of the Earth's surface the body is suspended by a string successively attached to various parts of it, there is one point in the body through which the straight line formed by extending the line of the string downwards always passes. This point is the *centre of gravity.*

[2] In a *solid*, the intermolecular forces are very strong and the molecules vary their relative positions only slightly; the spacing between molecules is comparable to their size. In a liquid, the intermolecular forces are less strong; molecules can move about readily (although their spacing is still comparable to their size) and they undergo frequent collisions. In a *gas*, the intermolecular forces are weak and the spacing is large compared with molecular dimensions, although it is still a very small distance (approximately 3×10^{-9} m (3 nm) for air at normal temperature and pressure).

Table 1.1. *Typical dimensions*

	Dimension (m)
Diameter of:	
an atomic nucleus	2×10^{-15}
an atom or gas molecule	6×10^{-10}
a polymer molecule	$\sim 10^{-8}$
Spacing of gas molecules	3×10^{-9}
Diameter of:	
a red blood cell	8×10^{-6}
a capillary	$4\text{–}10 \times 10^{-6}$
an artery	10^{-2}
the Earth	1.2×10^{7}
the Sun	1.4×10^{9}
the solar system	1.2×10^{13}
a galaxy	10^{20}
Spacing between galaxies	10^{22}

compared with those of the region of interest, but which still contain a very large number of molecules. With regard to the experiment, such an element effectively occupies a point, and can therefore be considered as a particle; with regard to molecular motions, however, it is very large, and its overall properties, like its velocity, or the density of the material in it, can be obtained by averaging over all the molecules which comprise it. We are thus able to ignore the random nature of molecular motion and treat materials as continuous. Newton's laws can now be applied to each element of the material (called a *fluid element*, or *fluid particle*, when the material is a fluid), and a precise and useful description of the motion as a whole will emerge.

In blood there are some very large molecules (e.g. lipoproteins, diameter about 3–5×10^{-8} m), and it flows in some very narrow tubes (some capillaries have a diameter as low as 4×10^{-6} m); but even so, the tube diameter is large compared with molecular dimensions. Thus blood plasma, for example, can be treated as a continuous fluid in the manner outlined above. Whole blood, however, cannot always be so treated, since it consists not only of plasma, but also of large numbers of cells which amount to about 45% of volume in normal man, and consist primarily of red blood cells (see Chapter 10). It would be convenient if the cells were small and numerous enough for their separate identity to be ignored, and their effect on the motion of whole blood, regarded as a continuous fluid, to be described in an average way. This is the case in large arteries (the diameter of the aorta, for example, is roughly 2000 times that of a red cell), but the diameter of a capillary is comparable to that of a red cell, and a description of flow in such small vessels must treat plasma and cells separately. To sum up, then, whole blood is effectively continuous in large vessels, but is not so in the microcirculation; plasma is continuous in both.

In Part I of this book, we shall develop the fundamental mechanics of continuous fluids and solids, although we must first outline Newton's laws of particle mechanics.

The mathematical symbols which appear are used solely as a form of shorthand, facilitating the precise expression of mechanical laws. They are all explained in words wherever they first appear, and a reader who knows some calculus will find much of the notation familiar.

2

Particle mechanics

Position

In order to describe the motion of a particle we must be able to describe accurately its position in space, which changes as the particle moves. To do this we suppose three straight lines to be drawn and fixed in space, all passing through a given point O, and each one perpendicular to the other two. The lines of intersection of two walls and the floor of a room are examples, with O in the corner of the room. If a fly were walking on the wall of the room (**Fig. 2.1**a), we could specify its position at any instant by recording its distance (say z) from the floor and its distance (say x) from a perpendicular wall. Similarly, if the fly were flying in the room, its position could be specified by recording its perpendicular distances from the three mutually perpendicular planes (the floor and two walls). And so it is with any point P whose position we wish to specify. Suppose that lines are drawn through P which intersect the three original lines at right angles at the points X, Y, Z (**Fig. 2.1**b). The

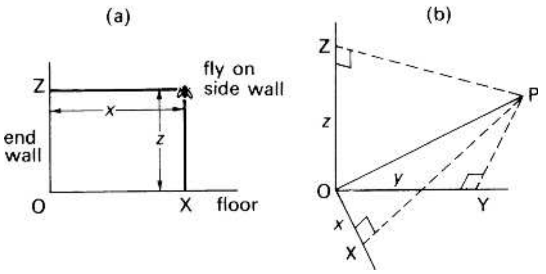

Fig. 2.1. (a) The position of a fly on the side wall of a room, specified by its distance x from the end wall and its distance z from the floor; x and z are its coordinates relative the the axes formed by the lines OX, OZ. (b) The position of a point P in three dimensions (a fly flying in a room) can be specified by its distances (x, y, z) from three mutually perpendicular planes (two walls and the floor). The coordinates of P are (x, y, z). The corner of the room O is the origin of coordinates.

three lengths OX (say x), OY(y), and OZ(z) then uniquely specify the position of P. These lengths are called the *coordinates* of P with respect to the three *axes* through O. The lines OX, OY and OZ are usually called the x-axis, y-axis and z-axis respectively. The total distance of P from O can be shown from Pythagoras' theorem to be equal to $\sqrt{x^2+y^2+z^2}$; this quantity is independent of the directions of the axes.

It is essential to remember two things implicit in this description. First, although the choice of the point O and the three axes is arbitrary, once it has been made it must be adhered to consistently. For instance, it would be hopeless to try to discuss the interaction between two particles if their positions were specified in relation to different corners of the room. We usually choose axes in the most convenient way – for example, if a particle is moving about on a flat plane (the fly on the wall), it is sensible to take one of the axes (say OY) perpendicular to that plane, so that y always remains constant and only two lengths, x and z, need be specified. Second, the units of length by which x, y and z are measured must be specified explicitly, and always used consistently. A length is not just a number, it is a quantity with dimension, and units are required to measure it. In this book, we shall usually use metres (m), centimetres ($1\,\text{cm} = 10^{-2}\,\text{m}$) or micrometres ($1\,\mu\text{m} = 10^{-6}\,\text{m}$); the whole question of units will be fully discussed later (Chapter 3).

Velocity

Another quantity of importance in describing the motion of a particle is its velocity, or the rate at which its position changes. Consider a particle moving along a straight line, OX (**Fig. 2.2**a), so that its position is specified by one coordinate, its distance x from O. If its coordinate at time t is x and its coordinate a little time later (t') is x', then the average velocity or speed of the particle in the interval of time from t to t' is $\upsilon = (x'-x)/(t'-t)$. This is well defined however short the time interval is; even if we let the interval become so short that $t'-t$, and hence $x'-x$, is vanishingly small, υ is still defined. The value to which υ tends as $t'-t$ tends to zero is the instantaneous velocity of the particle at time t; it is written

$$\upsilon = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.1}$$

evaluated at time t. The velocity is clearly negative if the particle is moving back towards O, so that x' is less than x.

The definition of $\mathrm{d}x/\mathrm{d}t$ can be understood graphically from (**Fig. 2.2**b). The upper graph shows how x varies with time t. Representative points P(x,t) and P$'(x',t')$ are marked. The quantity $(x'-x)/(t'-t)$ is the tangent of the angle ϕ between the line joining the two points and the time axis. This quantity is called the slope of that line. As $t'-t$ tends to zero, the point P$'$ moves towards the point P, and the line joining them approaches the tangent to the curve at P (shown in **Fig. 2.2**b as a dash–dot line).

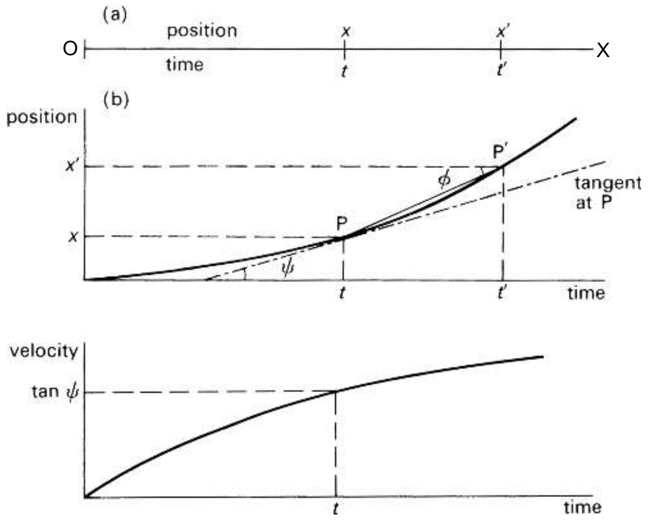

Fig. 2.2. (a) A particle moving along a line OX is at distance x from a fixed point O at time t, and at distance x' at a later time t'. The quantity $(x'-x)/(t'-t)$ is the average velocity of the particle during the time interval from t to t'. As this interval is made shorter, so that $t'-t$ tends to zero, $x'-x$ also becomes shorter, but the average velocity tends to a well-defined limit, υ. This is the velocity of the particle at time t. (b) The upper graph shows the distance x plotted against time (for a particular motion of the particle). The quantity $(x'-x)/(t'-t)$ is the slope of the line PP′ (and is equal to $\tan\phi$). As $t'-t$ tends to zero, this line becomes the tangent to the curve at the point P (broken line), whose slope is equal to υ $(=\tan\psi)$, the velocity of the particle at time t. The lower graph shows the corresponding plot of υ against t.

The quantity $\mathrm{d}x/\mathrm{d}t$, i.e. υ, is thus seen to be the slope of the tangent to the curve at P, and takes the value $\tan\psi$. The corresponding graph showing how υ varies with t is also presented in **Fig. 2.2**b.

The resolution of Zeno's paradox lies in the performance of this limiting procedure; without it there is no way of defining the instantaneous velocity of a particle in terms of its position at successive times. The procedure was in fact not thought of until the seventeenth century, when the calculus was first developed by Newton and Leibniz. In the notation of the calculus, the symbol $\mathrm{d}/\mathrm{d}t$ represents the rate of change of a quantity with time; in this example, all we mean by $\mathrm{d}x/\mathrm{d}t$ is the rate at which x changes with time t. The units of velocity must be taken to be consistent with the units chosen for distance and time. If distance is measured in metres and time in seconds, then velocity must be measured in metres per second ($\mathrm{m\,s^{-1}}$).

The above definition of velocity can readily be extended to situations where the particle is moving in three dimensions. If the fly already referred to were to fly from one corner of the room to the opposite corner, all its coordinates would change with time.

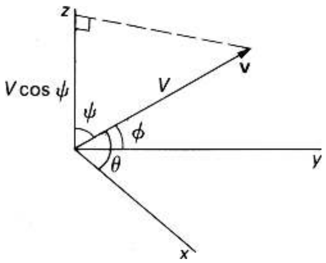

Fig. 2.3. The arrow marked **v** represents the velocity of a particle, with magnitude V and a certain direction. The component of the velocity in the direction of a coordinate axis is equal to $\upsilon_x = V\cos\theta,\ \upsilon_y = V\cos\phi,\ \upsilon_z = V\cos\psi$.

The specification of how they all vary would fully determine its motion. The point X of **Fig. 2.1** moves along the x-axis with velocity $\upsilon_x = \mathrm{d}x/\mathrm{d}t$, the point Y moves along the y-axis with velocity $\upsilon_y = \mathrm{d}y/\mathrm{d}t$ and the point Z moves along the z-axis with velocity $\upsilon_z = \mathrm{d}z/\mathrm{d}t$. The (three-dimensional) velocity of P is thus fully determined by the three quantities $(\upsilon_x, \upsilon_y, \upsilon_z)$, which are called the velocity components of P, in the x, y and z directions respectively. They clearly depend on the directions of the coordinate axes, but are independent of the position of the origin O. The total speed at which P is travelling, i.e. the component of its velocity along a line instantaneously parallel to the direction of motion, cannot depend on the directions of the axes. The speed, sometimes called the magnitude of the velocity, can be shown to be equal to $\sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2}$, which is a positive quantity even if some or all of υ_x, υ_y, υ_z are negative.

We can specify the velocity of a particle in three dimensions just as precisely by giving both its magnitude and its direction relative to any two of the coordinate axes (for example, if we know the direction of motion of the fly, and its total speed, then its velocity is fully determined). If the magnitude of the velocity is V, and the angles it makes with the x- and y-axes respectively are θ and ϕ (**Fig. 2.3**), then the components υ_x and υ_y are given by[1]

$$\upsilon_x = V\cos\theta, \qquad \upsilon_y = V\cos\phi.$$

The third component, υ_z, is then given by

$$\upsilon_z = \sqrt{V^2 - \upsilon_x^2 - \upsilon_y^2} = V\sqrt{1 - \cos^2\theta - \cos^2\phi},$$

which is also equal to $V\cos\psi$, where ψ is the angle between the direction of the velocity and the z-axis.

The velocity of a particle is an example of a physical quantity which has a certain magnitude and a certain direction. It exists independently of how we choose to

[1] The reader is assumed to be familiar with the elementary properties of sines and cosines.

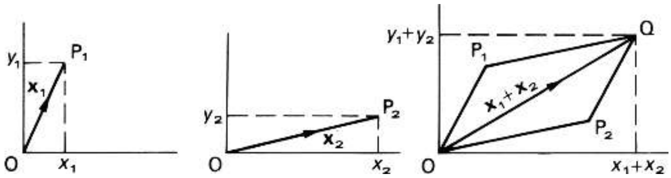

Fig. 2.4. These diagrams show the geometrical interpretation of Equation (2.3) for the addition of vectors, illustrated in two dimensions. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, represented by the lines OP_1 and OP_2, are added to form the vector $\mathbf{x}_1 + \mathbf{x}_2$, represented by the line OQ. The points O and Q are the opposite corners of a parallelogram two sides of which are OP_1 and OP_2.

measure it, although the value of the magnitude depends on the units used to measure it, and the specification of the direction depends on the orientation of the chosen coordinate axes. Such a quantity is called a *vector*, and vectors will be represented in this book by symbols in bold type. The velocity of a particle, for example, can then be written by the single symbol **v**. The quantities $(\upsilon_x, \upsilon_y, \upsilon_z)$ are the components of the vector **v**, and **v** can be regarded as equivalent to its three components taken together. We therefore often write $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z)$.

Another example of a vector quantity is the position of the particle P (**Fig. 2.1**b), which has magnitude equal to the length of OP ($\sqrt{x^2+y^2+z^2}$) and direction given by the cosines of the angles between OP and any two of OX, OY, OZ; alternatively, its components are the coordinates (x, y, z) themselves. This *position vector*, say **x**, is in fact a special type of vector, in that it does depend on the position of the origin O; all other vectors describing physical quantities, like velocity, are independent of the position of the origin. Vector notation, like the use of $\mathrm{d}/\mathrm{d}t$ for 'rate of change of', is just a convenient form of shorthand. From the definitions of υ_x, υ_y, υ_z as the rates of change of x, y, z (i.e. $\upsilon_x = \mathrm{d}x/\mathrm{d}t$, etc.), we can combine the two shorthand notations in an obvious way as follows:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}. \tag{2.2}$$

Velocity (a vector) is the rate of change of position (also a vector).

Vectors representing two quantities of the same type (for example, two velocities, or two position vectors) are added together by adding their components. Let $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2 = (x_2, y_2, z_2)$ be two such vectors; then

$$\mathbf{x}_1 + \mathbf{x}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2). \tag{2.3}$$

To see this, consider the situation in two dimensions (**Fig. 2.4**).

We add the vector $\mathbf{x}_1 = (x_1, y_1)$, representing the point P_1, to the vector $\mathbf{x}_2 = (x_2, y_2)$, representing the point P_2. For the geometric interpretation to remain consistent, the resulting vector should represent the point Q, whose coordinates are

(x_1+x_2, y_1+y_2). This is consistent with Equation (2.3). Alternatively, consider a projectile which, when fired from a fixed point, has velocity $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z)$. Now suppose that the point of firing is itself moving over the ground with velocity U in the x-direction, velocity vector $\mathbf{V} = (U, 0, 0)$. Then the velocity of the projectile relative to the ground is increased by an amount U in the x-direction, while its components in the y- and z-directions are unchanged, i.e. $\mathbf{v}+\mathbf{V} = (\upsilon_x + U, \upsilon_y, \upsilon_z)$.

Acceleration

In the same way as the velocity of a particle is defined as the rate of change of position, so the acceleration of the particle, defined as the rate of change of velocity, can also be written down. For motion along a line, the acceleration is $\mathrm{d}\upsilon/\mathrm{d}t$, which is the same as the slope of the tangent of the graph of υ against t (**Fig. 2.2**b). It too has three components, the rates of change of the three velocity components, and is also a vector, say **a**:

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\upsilon_x}{\mathrm{d}t}, \frac{\mathrm{d}\upsilon_y}{\mathrm{d}t}, \frac{\mathrm{d}\upsilon_z}{\mathrm{d}t}\right). \tag{2.4}$$

In the notation of calculus, if $u = \mathrm{d}x/\mathrm{d}t$, then $\mathrm{d}u/\mathrm{d}t$ can be written $\mathrm{d}^2x/\mathrm{d}t^2$, a useful shorthand for the rate of change of the rate of change of x. Thus we can write

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right). \tag{2.5}$$

The units of acceleration must, for consistency, be metres per second squared ($\mathrm{m\,s^{-2}}$).

It is perhaps a little difficult to grasp the precise definition of acceleration as a three-dimensional quantity. In one dimension, with the particle moving on a straight line OX, it is fairly easy: if the velocity v is increasing at a given moment, then the acceleration $a = \mathrm{d}v/\mathrm{d}t$ is positive; if v is decreasing a is negative. If the particle is moving back towards O with a positive value of x, then v is negative, but if it is at the same time slowing down, the acceleration a is positive. To make it clearer, **Fig. 2.5** shows graphs of x, υ and a against time t for a particle which starts from rest at O, accelerates up to a uniform speed which is maintained for some time, then decreases speed with constant negative acceleration until it has changed direction and is returning to O with the same uniform speed. Finally, it is slowed down and stopped at O again by the application of a positive acceleration. The direction of the acceleration (the sign of a) is independent of the direction of motion (the sign of u).

In two or three dimensions the direction of the acceleration is also independent of the direction of the velocity. Whenever the velocity is changing, either in magnitude or in direction, the particle experiences an acceleration. For example, suppose that a particle is travelling in a circle with constant speed, like a ball twirled on the end of a string or a satellite in its orbit round the Earth. In this case the *magnitude* of the