HOW TO FOLD IT

The Mathematics of Linkages, Origami, and Polyhedra

What do proteins and pop-up cards have in common? How is opening a grocery bag different from opening a gift box? How can you cut out the letters for a whole word all at once with one straight scissors cut? How many ways are there to flatten a cube?

You can answer these questions and more through the mathematics of folding and unfolding. From this book, you will discover new and old mathematical theorems by folding paper and find out how to reason toward proofs.

With the help of 200 color figures, author Joseph O'Rourke explains these fascinating folding problems starting from high school algebra and geometry and introducing more advanced concepts in tangible contexts as they arise. He shows how variations on these basic problems lead directly to the frontiers of current mathematical research and offers ten accessible unsolved problems for the enterprising reader. Before tackling these, you can test your skills on fifty exercises with complete solutions.

The book's web site, `http://www.howtofoldit.org`, has dynamic animations of many of the foldings and downloadable templates for readers to fold or cut out.

Joseph O'Rourke is Olin Professor and Chair of the Computer Science Department, a Professor of Mathematics, and Director of Arts and Technology at Smith College. His research is in computational geometry, developing algorithms for geometric computations. He has won several awards, including a Guggenheim Fellowship in 1987 and the NSF Director's Award for Distinguished Teaching Scholars in 2001. He has published more than 150 papers, more than 30 of which were coauthored with undergraduates. He has taught folding and unfolding to students in grade school, middle school, high school, college, and graduate school, and to teachers – of grade school, middle school, and high school – professors, and researchers. This is his sixth book.

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Preface

Cutting out the paper-doll figures below requires 64 straight scissors cuts if done without folding the paper. However, folding the paper along the dashed vertical creases lets you cut out all four people by just cutting one outline in the folded paper. Then you only have to do one-quarter of the work – 16 straight snips of the scissors rather than 64. Noticing that each figure is symmetric about a vertical line through the center of its octagonal head (humans have bilateral symmetry!) and additionally folding along that line permits cutting out all four people with eight straight scissors cuts, now through eight layers of paper. Wouldn't it be nice if there were a way to fold the pattern so that you could get away with a single straight slice of the scissors? Well, believe it or not, there is such a folding!

This beautiful "Fold and One-Cut" result is the topic of Chapter 5 in this book. We will see in that chapter that it is already not so straightforward to cut out a single irregular triangle in the center of a piece of paper with just one scissors cut. But understanding the trianHgle is a big step toward understanding how to cut out the four paper dolls. Folding the paper in preparation for cutting out a triangle reveals in its creases a theorem we all learned as teenagers (and most of us forgot!): The three angle bisectors of any triangle meet at a single point. Seeing the angle bisector creases converging at a point makes this abstract theorem (proved by Euclid) concrete and unforgettable.

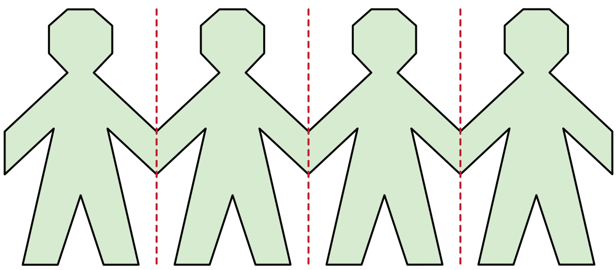

Figure 0.1. Paper-doll people for cut out.

This is the ideal at which I am aiming in this book. The nine chapters are unified by the notion of folding, but also unified by focusing on tangible constructions with rich mathematical content. I want you to be able to see the mathematical structure present in concrete, physical objects. The book is partitioned into three parts reflected in the title, which can be viewed loosely as concentrating on folding one-dimensional objects (linkages), two-dimensional paper (origami), and three-dimensional objects (polyhedra). (I will henceforth use 1D, 2D, and 3D as abbreviations.)

This book grew out of a monograph, *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*, aimed more at graduate students and researchers in computer science and in mathematics. Both my coauthor, Erik Demaine, and I have taught aspects of this material at various educational levels, from fifth-grade through high school, and found that the tangibility of the topics made them accessible through physical intuition. My goal in this book (which parallels the structure but not the content of *Geometric Folding Algorithms*) is to capitalize on the readers' physical intuition to introduce them to a variegated world of fascinating mathematics.

I assume only high-school mathematics: a little algebra, a little geometry, trigonometry only in a few marked exercises, no calculus or anything beyond. No computer science knowledge is presumed. Occasional boxed material explains technical terms and theorems that some readers will know but others will not (for example: vectors; the triangle inequality; convexity). Further terms are explained in the Glossary. Each chapter aims to reach one or a few mathematical gems. Because each topic is much larger than what I present, each chapter ends with an "Above & Beyond" section to explore more advanced results. I've avoided literature citations in the text, saving them for Chapter 10, "Further Reading."

Technical Terms and Symbols

I should explain two conventions from technical mathematical writing that may be unfamiliar to the reader. The first is that *technical terms* are italicized when introduced and defined, to alert the reader that a word or phrase is being given a special, usually technical meaning that may differ from its use in ordinary language. For example, in Chapter 1, I define the "shoulder" of a chain linkage in analogy with a human shoulder but with a specific meaning in the context of that chapter. The most important technical terms are gathered and defined in the Glossary. To avoid ambiguity, I <u>underline</u> for emphasis, reserving italics for technical definitions.

Second, there is a certain style of introducing symbols in mathematical writing to both shorten and make more precise the discussion. A typical example is, "Let x be a point on the polyhedron P in Figure 3." This means: Henceforth (for the duration of this discussion), we will use the symbol x to mean an arbitrary point on the polyhedron and the symbol P to mean the specific polyhedron illustrated in Figure 3. Sometimes the symbol introduction is flagged by "let … be," and sometimes it is implicit, as in the case of P above.

What Is a Proof?

Many students learn "two-column" proofs in high school, and then never take any higher-level mathematics courses, or, if they do, those courses do not contain proofs. For example, many calculus courses focus almost solely on the "calculating" aspects of calculus. Two-column proofs are the exception rather than the rule in mathematics. They may be the norm for *formal proofs*, where every step is justified by reference to an axiom or some previously established theorem. But most proofs in mathematics are a mixture of prose and symbols, often supplemented by reference to figures labeled with those symbols. A proof is something like a legal brief. It is intended to convince an appropriately prepared reader that a formal proof *could* be formulated, even though it rarely is. To achieve this, a proof must cover all cases, delve into every logical corner, and provide cogent reasons why the reader should "see" that all claims in the proof must be true. A proof is generally written for a particular audience, which defines what is "an appropriately prepared reader." The proofs that professional mathematicians write for one another would not be convincing for those without similar training.

This book contains many proofs, for I believe that proofs are the heart of mathematics. The audience member I am assuming for these proofs is an attentive reader who has taken (or is currently studying) standard high-school mathematics. I say "attentive" because I will describe a concept in one chapter and expect the reader to both master it and remember it in a later chapter (aided by a back-reference or the index). But the mastery will not require any background beyond high-school mathematics.

I also include several "proof sketches," which are a cut below a proof in that they do not pretend to handle every logically possible case, or to be stand-alone convincing. Proof sketches are intended to give the reader a feel for how a full proof might go. Often they leave out messy details and ask the reader to believe a claim that those details can all be worked out (and in fact, have all been worked out in the professional literature).

Theory Versus Applications

Despite the tangible aspect of folding, the material in this book focuses on the theoretical, as the emphasis on proofs indicates. A parallel and quite interesting book could be written that instead emphasized the applications of folding, only citing the underlying theory when appropriate. My tack has been nearly the obverse: I plunge into the theory and only cite the applications. I do this for two reasons. First, the underlying mathematics is beautiful in itself, a beauty that can only be fully appreciated by immersion in the details, and conveying this is one of my primary goals. Second, time and again, advances in mathematics seemingly divorced from reality have proved to have significant applications, sometimes much later. To cite just one example unrelated to folding, the Austrian mathematician Johann Radon invented in 1917 what is now known as the "Radon transform." Although his motivation was to extend the theoretical notion

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of integral from calculus to a special situation, the Radon transform is now used daily in hospitals the world over to reconstruct images taken by Computer-Aided Tomography (CAT) scanners. So the mathematical theory is both beautiful and often surprisingly useful.

Exercises

Each chapter contains a number of exercises, with answers in the back of the book (Answers to Exercises). I have partitioned them into three types: *Practice* – Questions to affirm basic understanding of the immediately prior material, often only requiring a bit of calculation; *Understanding* – Questions that require a thorough grasp of the preceding material, often applied to a slightly new situation; and *Challenge* – Problems that ask for substantive extensions and/or significant investments of time. The lines between these three classes are not sharp, nor are those lines the same for all readers. In any case, I encourage the reader to read each exercise, work out as many as circumstances permit, and in any case, to please look at the answers, which often enrich the material.

Templates on the Web

At a number of junctures in the book, particularly in the origami chapters, the reader is invited to cut out or fold a particular illustrated diagram. Each such diagram is available on the book's Web site, `http://howtofoldit.org/`. Each can be downloaded and printed. The Web site contains other useful supplementary information.

Open Problems

This book includes many unsolved problems, usually called *open problems* in mathematics. These are clear statements that have not yet been settled as either TRUE or FALSE by a proof. Sometimes researchers are convinced a hypothesis is TRUE even though they cannot prove it. In such a case, the hypothesis is designated as a *conjecture*. Some open problems have resisted all attempts over many years. However, most progress in mathematics occurs not by settling these long-unresolved problems but rather by answering recently posed questions. So I have included a number of new problems (a few concocted while writing this book), which may be open primarily through lack of attention. Rarely are the frontiers of mathematical knowledge accessible to the amateur, but one attractive aspect of the topic of folding is that many of its unsolved problems are accessible to the novice and might be solved by just the right clever idea. Please let me know if you crack one of them! They are listed in the Index under "open problems" for ease of access.