PART I

Linkages

Linkages are mechanisms built from stiff, inflexible bars, which we will call *rigid links*, connected at freely rotating joints. You may have a linkage on your desk similar to the one depicted in Figure I.1. Many machines contain linkages for particular functions. Every car contains a crankshaft, a mechanism for converting the linear motion induced from the sparked explosion of gasoline in a piston chamber to the rotary motion turning the drive shaft. We'll explore three linkages, each with a clean mathematical story to tell, and each related to developments on the frontiers of mathematics and computer science today: robot arms, pantographs, and fixed-angle chains. We'll analyze the "reachability region" for robot arms viewed as a linear chains of links. The pantograph is a mechanical copying and enlarging mechanism with myriad uses, especially during the industrial revolution. Fixed-angled chains are superficially similar to robot arms, but are primarily of interest as models of protein backbones.

Although here we are emphasizing the relevance of these linkages, our focus will be on the mathematics behind their operation.

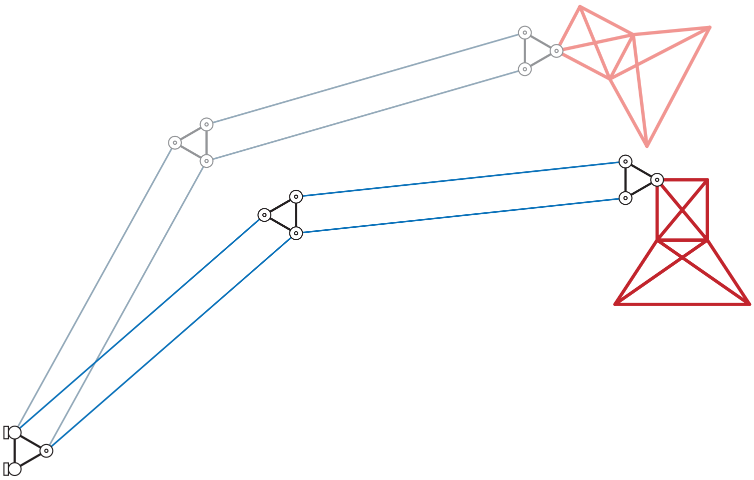

Figure I.1. A desk-lamp linkage. The linkage flexes at the circled joints, but is structurally rigid otherwise.

1 Robot Arms

Robot arms, despite their sophistication as machines, are particularly simple if you think of them as linkages. The arm in Figure 1.1(a), developed by a British robotics firm, is designed to apply adhesive tape to the edges of pieces of plate glass for protection. It has a fixed base (the *shoulder*) to which are attached three rigid links, corresponding roughly to upper arm, lower arm, and *hand*, or, in the technical jargon, the *end effector*. The rotation settings at the motorized joints determine the exact positioning of the hand as it performs its functions. The force dynamics and engineering aspects of robot arm design are quite interesting and challenging. However, we will focus on one simple question: determining what is called the *workspace* of the robot – the spots in space it can reach. We will pursue this question in almost absurd generality, permitting the arm to have an arbitrarily large number of links, each of an arbitrary length.

Model. First we need to reduce a complex physical robot arm to a simple mathematical model so that it can be analyzed. Typically, the initial abstraction chosen is crude, ignoring many physical details, and then, once analyzed, gradually made more realistic and complicated.

We reduce each robot arm piece to a straight-line *segment* of fixed length – a rigid link of mathematically zero thickness. Each joint motor is reduced to a mathematical point of zero extension joining the two *incident* links that it shares. So we have reduced the physicality of a real robot arm to segments and the endpoints of those segments; see Figure 1.1(b).

There are two more crucial physicalities to model: intersections and joint motions. Of course, no two distinct physical objects may occupy the same space at the same time, so the links should not be permitted to *intersect* – share points – except sharing the point at a common joint. However, we start our analysis with

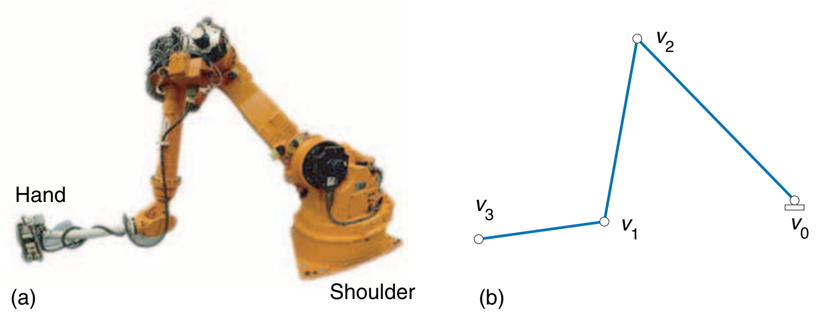

Figure 1.1. (a) A robot arm. (b) Arm modeled by linkage. v_0 labels the shoulder joint, and v_3 the hand.

the physically unrealistic assumption that intersections are ignored. Similarly, although most robot joints have physical constraints that prevent a full 360° rotation in two dimensions (2D), or free rotation in all directions in three dimensions (3D), we assume no joint constraint – so each is a *universal* joint, one that has total freedom of rotation. Later in exercises and in Chapter 3, we will constrain the joints.

So our mathematical model of a robot arm is a chain of n links, where n is some natural number $1, 2, 3, \ldots$, each a fixed-length segment of some prespecified length, connected by universal joints. For the robot arm in Figure 1.1, $n = 3$: 3 links, 3 joints (including the motorized shoulder). The hand/end effector is not a joint, just a link endpoint. Indeed, the number of links and the number of joints is always the same, n, under this convention of viewing the shoulder, but not the hand, as a joint.

Now the question is: Under this model, what is the totality of locations in 3D space that an n-link robot arm can reach? This set is called the *reachability region* of the arm.

At this point, we invite the reader to guess the answer that this chapter will soon establish more formally. Reasoning from your own shoulder-to-hand linkage may be misleading, because humans have definite (and complex) joint constraints. Perhaps it will help intuition to imagine a specific example. Suppose we have 3-link arm whose link lengths are 10, 5, and 3. What is the region of space that the hand endpoint can reach? Hint: It is not a sphere of radius 18!

Box 1.1: Theorem

In mathematics, the term *theorem* is used for a concise statement of a central result, whereas a *lemma* is a result that is a stepping-stone on the way to a theorem. A *corollary* is a near-immediate consequence of a theorem. Although we will not use the term, a *proposition* is often used for a relatively straightforward theorem.

1.1 Annulus

Rather than keep the reader in suspense, let us immediately move to the answer to this question, which we encapsulate in a theorem (see Box 1.1):

Theorem 1.1

The reachability region of an n-link robot arm is an annulus.

Now we should explain the term *annulus*. In 2D, an annulus is the region between two circles with the same center but different radii. Such circles are called *concentric*. The 3D analog, the region between two concentric spheres of different radii, is generally called a "spherical shell," but we opt to use "annulus" regardless of the dimension. See Figure 1.2(b). Right now we concentrate on 2D and consider 3D later (p. 19). For our 3-link example with link lengths 10, 5, and 3, the reachability region is an annulus with outer radius 18 and inner radius 2. That the inner radius is 2 is by no means obvious; it will be established later in Theorem 1.2.

There are two special cases that we further include under the term "annulus": (1) If the radii of both circles are equal, the region reduces to just that circle itself; (2) if one radius is zero, the region is the entire disk enclosed by the circle. A *circle* can be viewed as a rim "wire" whereas a *disk* includes the points inside the wire.

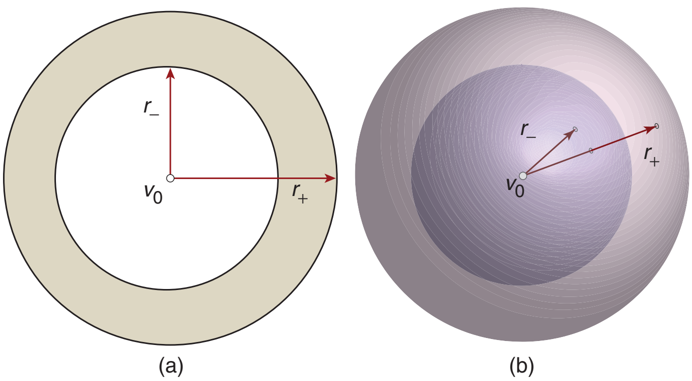

Figure 1.2. (a) 2D annulus: the region between two concentric circles. (b) 3D annulus (also known as a spherical shell): the region between two concentric spheres. Common centers are labeled v_0.

Box 1.2: Induction

Induction is a proof technique that can be used to establish that some claim is true for all numbers $n = 1, 2, 3, \ldots$. It is akin to climbing a ladder: If you know how to move from any one rung to the next, and you know how to reach the first rung, then you can climb to any rung, no matter how high. To reach the first rung, we only need prove the result holds for $n = 1$, the *base* of the induction. Moving from one rung to the next requires proving that if the theorem holds for $n-1$ (you've reached that rung), then the theorem holds for n, where n is an arbitrary natural number. Then the theorem must be true for all n, "by induction," as they say: From $n = 1$, we can reach $n = 2$, and from there we can reach $n = 3$, and so on.

Annulus Proof. The proof of Theorem 1.1 uses a method known as *induction*; see Box 1.2.

The base case is straightforward: A 1-link arm can reach the points on a circle, and by our definition, a circle is an annulus. Now we could jump immediately to the general case using induction. But let's look at $n = 2$ to build intuition; say the two link lengths are r_1 and r_2. This 2-link arm can reach all the points on a circle of radius r_2 centered on any of the points on a circle of radius r_1. Figure 1.3 illustrates the idea. Imagine sweeping the red r_2-circle around, centered on each point of the blue r_1-circle. The swept pink region R_2 is an annulus.

Let us now consider the general case, an n-link arm, $n > 1$. Following the induction paradigm, we assume that we have established the theorem for arms up to $n-1$ links. Then we know if we remove the last link of a given n-link arm

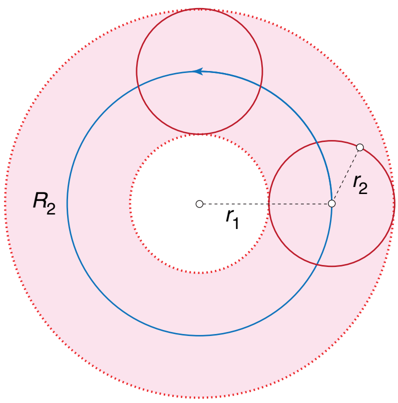

Figure 1.3. A 2-link arm can reach points in an annulus.

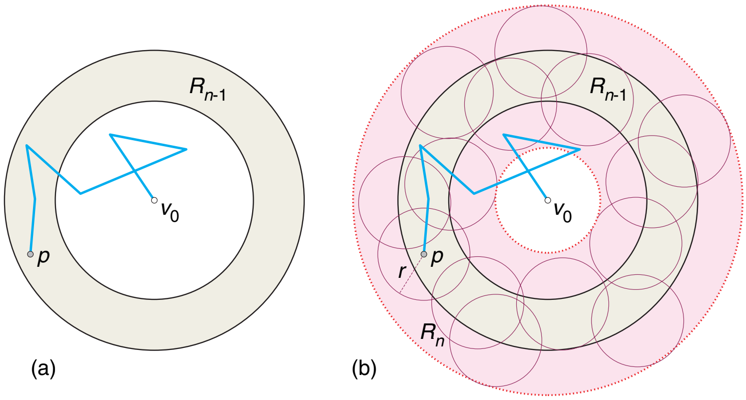

Figure 1.4. (a) R_{n-1} with one possible arm of $n-1$ links reaching a point p. (b) R_n is formed by adding the points on circles centered on every point p in R_{n-1}, with the radius r of these circles equal to the length of the last link of the arm.

(call it A_n), the shorter arm's reachability region is an annulus, because it has only $n-1$ links. (We have just employed the "induction hypothesis": $n-1$ link arms reach points in an annulus.) Let us call the shorter arm A_{n-1} and its region R_{n-1}. We seek to find R_n, the reachability region for A_n.

Let p be any point in R_{n-1}. We know that the hand of A_{n-1} can reach p, as in Figure 1.4(a). Now imagine adding the removed final link back to A_{n-1}. This permits A_n to reach all the points on a circle centered on p, where the circle's radius r is the length of that last link. So we can construct R_n by adding the points on a circle of radius r centered on every point p of R_{n-1}. See Figure 1.4(b).

Here I rely on the reader's intuition to see that R_n is again an annulus: Adding all these circles to an annulus results in a fatter annulus. Points p on the outer boundary of R_{n-1} reach out to a larger-radius circle bounding R_n, larger by r, and points on the inner boundary of R_{n-1} reach inward to a smaller-radius circle, smaller by r. Circles around points p in the interior of R_{n-1} fill out the remainder of the annulus. If r is enough to reach the center of R_{n-1}, then R_n becomes a disk, which we have defined as an annulus.

1.1.1 Radii

Our proof that the reachability region is an annulus does not directly yield the radii of the annulus. In particular, it would be useful to know under what conditions the reachability region is a disk, that is, when the hand can touch the shoulder. We now address this question.

Because the answer will depend on the arm's lengths, we will need some notation for those. Call the lengths of the n links $(\ell_1, \ell_2, \ldots, \ell_n)$, and call the outer and inner radii of the annulus r_+ and r_- respectively. The outer radius is easy: The furthest reach of the arm is achieved by straightening each joint, stretching the arm out straight. Recalling our 3-link example with lengths $(10, 5, 3)$,

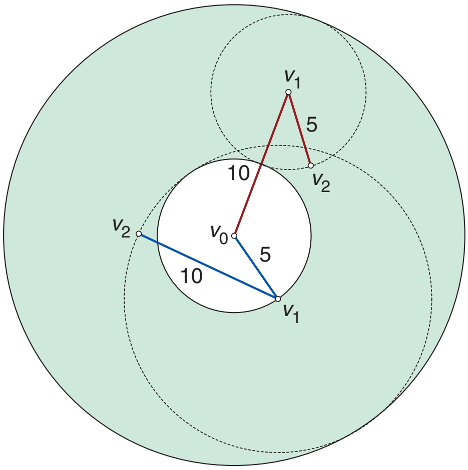

Figure 1.5. Annulus for the 2-link arm with lengths $(5,10)$ (red) is the same as for the arm with lengths $(10,5)$ (blue).

$r_+ = 10+5+3 = 18$. In general,

$$r_+ = \ell_1 + \ell_2 + \cdots + \ell_n \,.$$

Computing the inner radius r_- is less straightforward. A key idea that helps is hinted at by Figure 1.5, which shows that the reachability annulus for an arm consisting of two links of lengths 5 and 10 is independent of whether the longer or the shorter is the first link, incident to the shoulder. Somewhat surprisingly, this independence holds more generally:

Lemma 1.1

The reachability region of a robot arm is independent of the order of the link lengths: It only depends on the numerical values of those lengths, not the order in which they appear along the chain of links.

I will argue for this lemma before explaining its relevance to computing r_-. Let v_0 be the location of the shoulder joint of the arm, and $v_1, v_2, \ldots, v_{n-1}, v_n$ the positions of the remaining joints, or, as they are commonly known in geometry, the *vertices* of the chain. (The singular is *vertex*.) The last vertex v_n is the position of the hand, not considered a joint (because there is nothing beyond that it joins). In any particular configuration of the arm, the vertices are at particular points in the plane. We take v_0 to be the origin of the coordinate system in which we express the points: $v_0 = (0,0)$.

Box 1.3: Vectors

We illustrate the notion of a vector with the 3-link arm shown in Figure 1.6(a), whose shoulder is at $v_0 = (0,0)$ and whose vertices are located at $v_1 = (1,1)$, $v_2 = (1,0)$, and $v_3 = (0,3)$, with the shoulder at the origin $v_0 = (0,0)$. The lengths of the links are $\ell_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\ell_2 = 1$, and $\ell_3 = \sqrt{1^2 + 3^2} = \sqrt{10}$. We can view each successive vertex as displaced from the previous one. So v_2 is obtained from v_1 by moving vertically down one unit, and v_3 is obtained from v_2 by one step left horizontally and three up vertically. These displacements are *vectors*, and can be computed by subtracting the points coordinate by coordinate. We will use uppercase letters with over-arrows to indicate vectors. So,

$$\vec{V_1} = v_1 - v_0 = (1,1) - (0,0) = (1,1)$$

corresponds to moving right and up 1,

$$\vec{V_2} = v_2 - v_1 = (1,0) - (1,1) = (0,-1)$$

corresponds to moving down 1, and

$$\vec{V_3} = v_3 - v_2 = (0,3) - (1,0) = (-1,3)$$

corresponds to 1 left, 3 up. Because we chose $v_0 = (0,0)$, $\vec{V_1} = v_1 - v_0$ is the same as v_1: $(1,1)$. The length of these vectors is exactly the link length which they "span," for example, the length of $\vec{V_3}$ is $\sqrt{10}$.

There is a certain ambiguity when we represent a point by its coordinates and a vector by its coordinate displacements, for they both look the same as pairs of numbers: Thus the point v_1 has the same coordinate representation as the vector $\vec{V_1}$. But a point is a location in the plane, whereas a vector is a displacement in the plane. Every point in the plane can be viewed as a displacement from the origin – a viewpoint that is often convenient.

Two vectors are added by adding their displacements coordinate by coordinate. So the sum of the vectors $(1,1)$ and $(0,-1)$ is $(1,0)$, which, not surprisingly, is v_2:

$$\vec{V_1} + \vec{V_2} = (v_1 - v_0) + (v_2 - v_1) = v_2 - v_0$$

which is v_2 because $v_0 = (0,0)$.

The key to the proof of this lemma is to think of the vertices of the joints as reached by a series of *vector* displacements from the shoulder. Vectors are an important concept we will use in several chapters; see Box 1.3. Let us represent the vector displacement between adjacent vertices with the symbol $\vec{V_i}$, with $\vec{V_i} = v_i - v_{i-1}$, where the subscript i can take on any integer value between 1

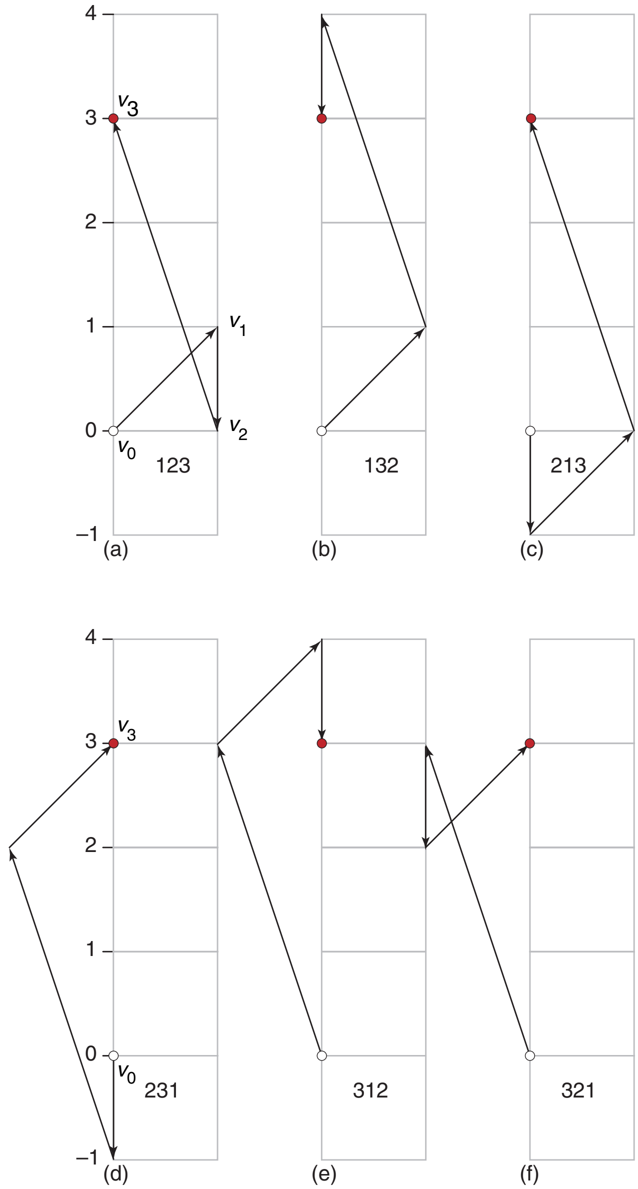

Figure 1.6. A 3-link arm reaching from $v_0 = (0, 0)$ (white circle) to $v_3 = (0, 3)$ (red circle). All six possible permutations (indicated below each figure) of adding the three vector displacements all reach to $(0, 3)$.