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Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
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Introduction

The purpose of this opening chapter is threefold: to introduce the Hubbard model, to discuss its origin and significance and to give a brief summary of its history. Rather than beginning with more general and historical considerations we will start with a concrete albeit somewhat technical discussion of how the Hubbard model arises as an effective description of electronic degrees of freedom in solids.

1.1 On the origin of the Hubbard model

The Hubbard model is named after John Hubbard, who in a series of influential articles [201–206] introduced[1] the Hamiltonian in order to model electronic correlations in narrow energy bands and proposed a number of approximate treatments of the associated many-body problem. Our following discussion of how the Hubbard Hamiltonian arises in an approximate description of interacting electrons in a solid loosely parallels Hubbard's original work. We will assume that the reader is familiar with the basic concepts of solid state theory (see e.g. [25, 509]) and with the formalism of second quantization (e.g. [283]). For further reading we refer to the original literature [188, 201, 233] and to the monographs [27, 158, 498].

A solid consists of ions and electrons condensed in a three-dimensional crystalline structure. Since the ions are much heavier than the electrons, it is often a good phenomenological starting point for the exploration of the electronic properties of solids to think of the ions as forming a *static* lattice.[2] In this approximation the dynamics of the electrons is governed by the Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V_I(\mathbf{x}_i) \right) + \sum_{1 \le i < j \le N} V_C(\mathbf{x}_i - \mathbf{x}_j) \,, \tag{1.1}$$

where N is the number of electrons, $V_I(\mathbf{x})$ is the periodic potential of the ions and

$$V_C(\mathbf{x}) = \frac{e^2}{\| \mathbf{x} \|} \tag{1.2}$$

is the Coulomb repulsion among the electrons.

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[1] The Hubbard model was independently introduced by Gutzwiller [188] and Kanamori [233] around the same time.
[2] This can be further justified within the Born-Oppenheimer approximation [68].

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In spite of the drastic approximation we made by assuming a static lattice the Hamiltonian (1.1) is far too complicated to be solved exactly. It still bears all the difficulties of a generic many-body system. Much of the success of solid state theory derives from efficient 'mean-field' one-particle approximations to (1.1). On a technical level these approximations are based on adding an auxiliary potential $V_A(\mathbf{x})$ to the one-particle piece of the Hamiltonian (1.1) and then subtracting it again in the two-body part, i.e. we may write

$$H = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{x}_i) \right) + \sum_{1 \le i < j \le N} U(\mathbf{x}_i, \mathbf{x}_j)\,, \tag{1.3}$$

where we introduced effective one- and two-body potentials $V(\mathbf{x})$ and $U(\mathbf{x}, \mathbf{y})$ as

$$V(\mathbf{x}) = V_I(\mathbf{x}) + V_A(\mathbf{x})\,, \tag{1.4a}$$

$$U(\mathbf{x}, \mathbf{y}) = V_C(\mathbf{x} - \mathbf{y}) - \tfrac{1}{N-1} \big(V_A(\mathbf{x}) + V_A(\mathbf{y}) \big)\,. \tag{1.4b}$$

Mean-field approximations to H amount to simply setting $U(\mathbf{x}, \mathbf{y})$ equal to zero. In order for this to be sensible, the auxiliary potential needs to be chosen in such a way, that the matrix elements of the effective two-body potential $U(\mathbf{x}, \mathbf{y})$ between the eigenstates of the one-particle Hamiltonian

$$h_1(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{1.5}$$

become small. Even in circumstances when this cannot be achieved, the two-body interaction $U(\mathbf{x}, \mathbf{y})$ may still be considerably reduced in range and magnitude compared to the full Coulomb interaction $V_C(\mathbf{x} - \mathbf{y})$.

The physical idea behind the introduction of the auxiliary potential $V_A(\mathbf{x})$ may be formulated as follows. Let us assume we have a large number N of electrons in the ground state Ψ_0 of the Hamiltonian H. If we insert an additional electron locally into the system, what potential does it feel? Superimposed on the periodic potential of the ions it feels the electro-static potential which stems from the ground state density $|\Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N)|^2$ of the other electrons. This potential is periodic with the same periods as the ionic potential. It is, however, of the opposite sign and therefore screens the attractive interaction of the ions. Of course, this picture is only approximately correct because the additional electron itself causes a change in the ground state density. Still, we may imagine that (again because of the screening) the effect of the additional electron is only local and therefore small.

We now wish to 'second-quantize' the Hamiltonian (1.3) in a suitable basis of states. In order to construct this basis we consider eigenstates of the one-particle Hamiltonian h_1. Since the one-body potential $V(\mathbf{x})$ in (1.4a) is periodic, the eigenfunctions of h_1 are Bloch functions (see e.g. [25, 509]), i.e., they are of the form

$$\varphi_{\alpha\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} u_{\alpha\mathbf{k}}(\mathbf{x})\,. \tag{1.6}$$

Here $u_{\alpha\mathbf{k}}(\mathbf{x})$ has the periodicity of the lattice, $\mathbf{k}$ is the quasi momentum and α the band index. The quasi momentum vector $\mathbf{k}$ runs over the first Brillouin zone. Being eigenfunctions of

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the one-particle Hamiltonian h_1,

$$h_1 \varphi_{\alpha\mathbf{k}}(\mathbf{x}) = \varepsilon_{\alpha\mathbf{k}} \varphi_{\alpha\mathbf{k}}(\mathbf{x})\,, \tag{1.7}$$

the functions $\varphi_{\alpha\mathbf{k}}(\mathbf{x})$ constitute a basis of one-particle states.

A complementary one-particle basis is provided by the Wannier functions [25, 509] $\phi_\alpha(\mathbf{x} - \mathbf{R}_i)$, where $\mathbf{R}_i$ is a lattice vector and $\phi_\alpha(\mathbf{x})$ is defined as

$$\phi_\alpha(\mathbf{x}) = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \varphi_{\alpha\mathbf{k}}(\mathbf{x})\,. \tag{1.8}$$

Here L denotes the number of ions. The Wannier functions $\phi_\alpha(\mathbf{x} - \mathbf{R}_i)$ are centred around $\mathbf{R}_i$. They are lattice analogues of atomic wave functions and have the advantage of being mutually orthogonal for different band and site indices α and i. The Bloch functions are expressed in terms of the Wannier functions by means of Fourier inversion,

$$\varphi_{\alpha\mathbf{k}}(\mathbf{x}) = \frac{1}{\sqrt{L}} \sum_{i} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}_i} \phi_\alpha(\mathbf{x} - \mathbf{R}_i)\,. \tag{1.9}$$

Let us introduce creation operators $c_{\alpha\mathbf{k},a}^\dagger$ of electrons of spin a in Bloch states $\varphi_{\alpha\mathbf{k}}(\mathbf{x})$. We further introduce their Fourier transforms

$$c_{\alpha i,a}^\dagger = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}_i} c_{\alpha\mathbf{k},a}^\dagger\,. \tag{1.10}$$

Then, using (1.9), we may express the field operator, which creates an electron of spin a at position $\mathbf{x}$, in two different ways,

$$\Psi_a^\dagger(\mathbf{x}) = \sum_{\alpha\mathbf{k}} \varphi_{\alpha\mathbf{k}}^*(\mathbf{x})\, c_{\alpha\mathbf{k},a}^\dagger = \sum_{\alpha i} \phi_\alpha^*(\mathbf{x} - \mathbf{R}_i)\, c_{\alpha i,a}^\dagger\,. \tag{1.11}$$

Here the asterisk denotes complex conjugation.

Finally, the general formula (see [283]) relating first and second quantized formalisms

$$\begin{aligned} H = {} & \sum_{a=\uparrow,\downarrow} \int \mathrm{d}x^3\ \Psi_a^\dagger(\mathbf{x})\, h_1 \Psi_a(\mathbf{x}) \\ & + \tfrac{1}{2} \sum_{a,b=\uparrow,\downarrow} \int \mathrm{d}x^3 \mathrm{d}y^3\ \Psi_a^\dagger(\mathbf{x}) \Psi_b^\dagger(\mathbf{y}) U(\mathbf{x},\mathbf{y}) \Psi_b(\mathbf{y}) \Psi_a(\mathbf{x})\,, \end{aligned} \tag{1.12}$$

enables us to express the Hamiltonian (1.1) in second quantized form in the basis of Wannier states,

$$H = \sum_{\alpha,i,j,a} t_{ij}^\alpha\, c_{\alpha i,a}^\dagger\, c_{\alpha j,a} + \tfrac{1}{2} \sum_{\substack{\alpha,\beta,\gamma,\delta \\ i,j,k,l}} \sum_{a,b} U_{ijkl}^{\alpha\beta\gamma\delta}\, c_{\alpha i,a}^\dagger\, c_{\beta j,b}^\dagger\, c_{\gamma k,b}\, c_{\delta l,a}\,. \tag{1.13}$$

Here the hopping matrix elements t_{ij}^α are given by

$$t_{ij}^\alpha = \int \mathrm{d}x^3\ \phi_\alpha^*(\mathbf{x} - \mathbf{R}_i)\, h_1 \phi_\alpha(\mathbf{x} - \mathbf{R}_j) = \frac{1}{L} \sum_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{R}_i - \mathbf{R}_j)} \varepsilon_{\alpha\mathbf{k}}\,. \tag{1.14}$$

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Similarly, the interaction parameters $U_{ijkl}^{\alpha\beta\gamma\delta}$ are expressed as 'overlap integrals'

$$U_{ijkl}^{\alpha\beta\gamma\delta} = \int \mathrm{d}x^3 \mathrm{d}y^3 \; \phi_\alpha^*(\mathbf{x} - \mathbf{R}_i)\phi_\beta^*(\mathbf{y} - \mathbf{R}_j)U(\mathbf{x}, \mathbf{y})\phi_\gamma(\mathbf{y} - \mathbf{R}_k)\phi_\delta(\mathbf{x} - \mathbf{R}_l) \,. \quad (1.15)$$

We note that H in equation (1.13) is still completely equivalent to the first quantized Hamiltonian (1.1). An optimal choice of the Wannier functions $\phi_\alpha(\mathbf{x})$ through an optimal choice of the auxiliary potential $V_A(\mathbf{x})$ minimizes the influence of the mutual Coulomb interaction. When the interaction parameters are small compared to the hopping matrix elements, they can be set equal to zero in a first approximation, and can later be taken into account by perturbation theory. This is the realm of band theory.

The Hubbard model is obtained from (1.13) when the interaction parameters are no longer negligible, but their range is still very small, i.e., when the intra-atomic Coulomb interaction $U_{iiii}^{\alpha\beta\gamma\delta}$ is large compared to the inter atomic interaction parameters and, at the same time, cannot be neglected compared to the hopping matrix elements. This situation is believed to be characteristic for transition and rare earth metals.

When the Fermi surface lies inside a single conduction band, say $\alpha = 1$, it is sometimes justified to 'project' the multi-band Hamiltonian onto an effective one-band model. Let us imagine a situation where the interband interactions are weak and at the same time all bands except the $\alpha = 1$ conduction band are far away from the Fermi level. As long as we are interested only in energies in the vicinity of the Fermi level, the main effect of the high energy bands is to change the hopping and interaction parameters of the electrons in the conduction band. Then we may replace the multi-band Hamiltonian (1.13) by a one-band model with effective parameters t_{ij} and U

$$H = \sum_{ij} t_{ij} c_{i,a}^\dagger c_{j,a} + \tfrac{U}{2} \sum_i c_{i,a}^\dagger c_{i,b}^\dagger c_{i,b} c_{i,a} \,. \quad (1.16)$$

Whereas the hopping matrix elements can usually be determined accurately in the framework of density-functional theory (see e.g. [106]), the effective interaction parameter U is much more difficult to estimate and is perhaps best fixed by comparing theoretical predictions to experimental results.

It has to be said that the Hamiltonian (1.16) is not expected to describe the transition or rare earth metals quantitatively, since the interaction between overlapping bands is important in both cases. The Hamiltonian (1.16) is most appropriately regarded as an effective Hamiltonian that is believed to capture, at least qualitatively, some of the electronic features of the transition metals.

A further simplification of the Hamiltonian, which is compatible with the assumption that the Wannier functions $\phi_\alpha(\mathbf{x} - \mathbf{R}_i)$ are strongly localized around $\mathbf{R}_i$ is the tight-binding approximation, where one retains only hopping matrix elements between nearest neighbours. Then, upon introducing the particle number operators $n_{i\uparrow} = c_{i\uparrow}^\dagger c_{i\uparrow}$ and $n_{i\downarrow} = c_{i\downarrow}^\dagger c_{i\downarrow}$,

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the Hamiltonian (1.16) reduces to

$$H = -t \sum_{\langle i,j \rangle} c_{i,a}^{\dagger} c_{j,a} + U \sum_{i} n_{i\uparrow} n_{i\downarrow} . \tag{1.17}$$

Here the symbol $\langle i, j \rangle$ denotes summation over ordered pairs of nearest neighbours. We have assumed isotropic hopping of strength $-t$ between nearest neighbours and have suppressed the on-site terms t_{ii}, since they may be absorbed into the chemical potential in a grand canonical description of the model.

The one-dimensional version of the Hamiltonian (1.17) is easily identified with the Hubbard Hamiltonian (2.1) of Chapter 2 which is the actual starting point of this book. Its peculiar charm is certainly due to the fact that, in spite of its simplicity, it cannot be reduced to a one-particle theory and for this reason shows a rich spectrum of physical phenomena.

1.2 The Hubbard model – a paradigm in condensed matter physics

One of the most successful descriptions of electrons in solids is band theory. It is based on reducing many-body interactions to an effective one-body description, i.e., on neglecting the two-body potential $U(\mathbf{x}, \mathbf{y})$ in equation (1.4b) or equivalently the interaction parameters $U_{ijkl}^{\alpha\beta\gamma\delta}$ in equation (1.13). However, there are various situations of physical interest where band theory fails by construction. Arguably the most prominent example are Mott insulators: these have an odd number of valence electrons per elementary cell and yet are insulating, in contradiction with predictions of band theory.

One of the main motivations for studying the Hubbard model is that it is the simplest generalization beyond the band theory description of solids, yet still appears to capture the gross physical features of many systems characterized by more general interaction parameters in (1.13). The Hubbard model has been used in attempts to describe

(i) the electronic properties of solids with narrow bands,
(ii) band magnetism in iron, cobalt, nickel,
(iii) the Mott metal-insulator transition,
(iv) electronic properties of high-T_{c} cuprates in the normal state.

Despite its apparent simplicity, no fully consistent treatment of the Hubbard model is available in general. However, there are two cases in which one is more fortunate and many properties are calculable, namely the extremes of lattice coordination numbers two and infinity.[3] One might naively expect that the latter case can be easily understood by means of a mean-field approximation. Surprisingly, there is a particular way [325] of performing the limit of infinite lattice 'dimension' $D \to \infty$, in which the behaviour of the Hubbard model does not become mean-field like, but the model remains tractable. A striking result obtained in this approach is an understanding of the Mott transition between a paramagnetic metal and a correlated insulator. For details we refer the interested reader to the review article [163] and to the monograph [158].

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[3] Notice that a few rigorous results [294, 452] also hold in the general case of arbitrary lattice dimension.

Here we are concerned with the first case, which corresponds to a one-dimensional lattice. The one-dimensional Hubbard model has a distinctive feature: it is 'integrable'. This essentially means that many physical properties can be determined exactly.[4] Integrable models are rather special and occur mostly in $D = 1$. Integrability is a fragile property: adding extra terms to an integrable Hamiltonian will in general break it. This fact is a frequent source of criticism. However, often the physics of a given problem is more robust than the mathematics: adding small perturbations to a given Hamiltonian does not necessarily lead to any dramatic changes in the physical properties. This point can be made more precise in the context of universality classes of critical behaviour (see e.g. [76]). By the same token the 1D Hubbard model may yet prove to be directly relevant for the description of experiments.

In our view, the central importance of integrable models rests with the fact that they constitute paradigms of diverse physical phenomena, which can be understood and characterized in their entirety. They allow us to study many-body physics beyond the restrictions of perturbation theory or intuitive non-systematic approximations. Their analysis permits us to develop an intuitive understanding of 'non-perturbative' effects. Last but not least, integrable models provide benchmarks for the development of approximate and numerical methods.

1.2.1 Integrable models

The history of exactly solvable many-body quantum systems traces back to H. Bethe's 1931 article [60] on the spin-$\frac{1}{2}$ Heisenberg chain in the early days of Quantum Theory. Bethe constructed the many-body wave functions and reduced the problem of calculating the spectrum of the Hamiltonian to solving a set of N coupled algebraic equations ('Bethe ansatz equations'), where N is the number of overturned spins. In this way a problem of exponential complexity is reduced to one of polynomial complexity. Bethe's work provides an explicit answer for the ground state properties and excitations of the ferromagnetic Heisenberg model. The energy per lattice site in the antiferromagnetic ground state was calculated by L. Hulthén in [207]. Hulthén recognized that in the thermodynamic limit the ground state can be characterized by the solution of a linear integral equation.

At the time Bethe's work was considered to be a fascinating but mostly academic exercise and it was hoped that it might serve as a stepping stone on the path to a solution of Heisenberg models on two- and three-dimensional lattices. Since then, Bethe's work has remained a constant source of inspiration for generations of researchers in theoretical and mathematical physics. It marked the beginning of a new branch of mathematical physics, the theory of exactly solvable quantum systems.

The next milestone was reached in 1944 [350] with L. Onsager's solution of the two-dimensional Ising model, which is based on an infinite-dimensional symmetry algebra ('Onsager Algebra') and a transfer matrix approach. Furthermore, the 'star-triangle relation', which played a crucial role in many subsequent developments, was mentioned for the first

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[4] A precise definition of integrability is given in Chapter 12.

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time. Onsager's work had a lasting impact on the microscopic foundations of the general theory of phase transitions. It established that singularities in the free energy can be obtained by a direct calculation of the partition function of a microscopic model. Furthermore, the solution demonstrated that critical exponents need not be mean-field like.

New applications of Bethe's ansatz were discovered during the 1960s, starting with the work [296] of E. H. Lieb and W. Liniger on the Bose gas with delta-function interactions. The extension of Bethe's ansatz to problems in statistical mechanics was achieved by E. H. Lieb in his solution of three archetypical cases of the six-vertex model (ice, KDP, F) [290–292]. The general case was solved shortly after by B. Sutherland [422] (for a review see [299]).

The generalization of Bethe's ansatz to models with internal degrees of freedom like spin proved to be very hard, because scattering involves changes of the internal states of the scatterers. This problem was eventually solved by C. N. Yang [493] and M. Gaudin [154] by means of what is nowadays called 'nested Bethe ansatz'. The condition for the applicability of the nested Bethe ansatz is the consistent factorization of multi-particle scattering processes into two-particle ones. Consistency requires the two-particle scattering matrices to fulfill certain algebraic equations, the 'Yang-Baxter Equations'.

In 1969 C. N. Yang and C. P. Yang [496] showed that Bethe's ansatz allows for the calculation of finite temperature properties of the delta-function Bose gas. This astonishing result is the first exact treatment of the thermodynamics of an interacting many-body quantum system.

Starting from the observation that the eigenstates of the transfer matrix of the six-vertex model are independent of one of the parameters, R. J. Baxter realized that there must be an entire family of commuting transfer matrices. He discovered a simple explanation of this remarkable fact by showing that it follows from ternary relations for the local Boltzmann weights [45]. These relations are a sufficient condition for the solvability of the model and are identical to the Yang-Baxter Equations obtained previously by Yang in his construction of the nested Bethe ansatz. Using his insights, Baxter realized the equivalence of the Yang-Baxter Equations to Onsager's star-triangle relations. He also discovered that the Boltzmann weights of an eight-vertex model satisfy the Yang-Baxter Equations, establishing solvability in the sense of the existence of a family of commuting transfer matrices [43,44]. Interestingly the model cannot be solved by Bethe's ansatz. However, Baxter managed to develop novel methods for the calculation of partition functions as well as one-point functions [45].

The role of the Yang-Baxter Equations as the defining structure of integrable models was emphasized by L. D. Faddeev, E. K. Sklyanin and L. Takhtajan and other members of the St Petersburg branch of the Stekhlov Mathematical Institute. They established a relation between quantum many-body models solved by Bethe's ansatz and classical integrable evolution equations [404, 410, 411]. Building on this connection, they initiated a systematic search for solutions of the Yang-Baxter Equations [274, 276] and developed a programme for the solution of integrable models they called the 'Quantum Inverse Scattering Method'. An important element of this method is the algebraization of the construction of eigenstates of the transfer matrix [132, 445]. The developments initiated by the Stekhlov group culminated in the advent of 'Quantum Groups' [107, 225].

1.2.2 *Bethe ansatz solution of the Hubbard model*

The history of the one-dimensional Hubbard model as an exactly solvable model began in 1968 with E. H. Lieb and F. Y. Wu's article [298]. Lieb and Wu discovered that Bethe's ansatz can be applied to the Hubbard model and reduced the spectral problem of the Hamiltonian to solving a set of algebraic equations, nowadays known as the Lieb-Wu equations (see Chapter 3). They succeeded in calculating the ground state energy and demonstrated that the Hubbard model undergoes a Mott metal-insulator transition at half filling (one electron per site) with critical interaction strength $U = 0$ (Chapter 6).[5]

In the 35 years since Lieb and Wu's fundamental work appeared in print there have been hundreds of publications on the subject. It is clearly an impossible feat to do all of them justice within the confines of this short introduction. Hence we will constrain the following discussion to a small selection of works, which in our very personal and subjective view are of particular importance.

In 1972 M. Takahashi [435] proposed a classification of the solutions of the Lieb-Wu equations in terms of a 'string hypothesis' (see Chapter 4). He employed this hypothesis to replace the Lieb-Wu equations by simpler ones and then proceeded to derive a set of non-linear integral equations, which determine the Gibbs free energy of the Hubbard model (Chapter 5). These integral equations are known as thermodynamic Bethe ansatz (TBA) equations. Solving them in the limit of small temperatures Takahashi calculated the specific heat [436]. Later on a more complete picture of the thermodynamics of the Hubbard model was obtained from numerical solutions of the TBA equations [240, 469].

In fact, Takahashi's equations, in conjunction with the TBA equations, can be used to calculate any physical quantity that pertains to the energy spectrum of the Hubbard model. In particular, the dispersion curves of all elementary excitations can be obtained from the TBA equations in the limit $T \to 0$ [95]. Constraints on the quantum numbers in Takahashi's equations imply certain selection rules that determine the allowed combinations of elementary excitations and therefore the physical excitation spectrum [95]. Historically, the pioneering works in which ground state properties [298, 396, 429, 492] and the excitation spectrum [88, 258, 352, 481, 482, 485, 486]) were determined followed a different approach. In our view the thermodynamic Bethe ansatz is perhaps the most systematic approach for studying the ground state and the physical excitation spectrum of the one-dimensional Hubbard model and will serve as the basis of the corresponding Chapters 6 and 7 of this book.

Takahashi's equations may also serve as starting point for the calculation of the scattering matrix of the elementary excitations. For the half-filled Hubbard model in vanishing magnetic field the S-matrix was calculated in [120, 121]. It was shown that the excitation spectrum at half filling is given by scattering states of four elementary excitations: holon and antiholon with spin 0 and charge $\pm e$ and charge neutral spinons with spin up or down respectively. This is remarkable, since away from half filling, or at finite magnetic field, the number of elementary excitations is infinite [95]. It was further shown in [120, 121] that

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[5] The ground state of the half-filled Hubbard model is metallic for $U = 0$ but Mott-insulating for all $U > 0$.

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the four particles can only be excited in SO(4) multiplets (for the SO(4) symmetry of the model see Chapters 2 and 3).

A new chapter in the analysis of integrable models was opened with the advent of Conformal Field Theory [51]. In conformally invariant one-dimensional quantum systems the critical exponents governing the power-law decay of correlation functions are directly related to the energy levels in a large finite volume [6, 62, 74, 75]. Fortuitously in integrable theories the finite-size corrections to the energies of the ground state and low lying excited states can be determined from the Bethe ansatz [470]. The relations derived in [6,62,75] have been applied to numerous integrable models, including the attractive Hubbard model [65], which are conformally invariant in the low energy limit. Systematic studies of the finite size corrections in the spectrum of the Hubbard model were performed in the late eighties starting with work on the half-filled case [489]. The generic finite size spectrum away from half filling in finite magnetic fields was eventually obtained by F. Woynarovich [487]. In [140, 141] the aforementioned relation between the finite-size spectrum and the asymptotic behaviour of correlation functions was extended to models with several critical degrees of freedom and then utilized to calculate the critical exponents of general two-point correlation functions for the repulsive Hubbard model using Woynarovich's results (see Chapter 9). This was a breakthrough in the understanding of correlations in interacting one-dimensional quantum systems. The method has subsequently been applied to many other integrable models.

When considering a finite volume, an issue arises which we have hitherto ignored, namely the boundary conditions imposed on the system. In Lieb and Wu's work periodic boundary conditions were chosen. In 1985, the Hubbard model with reflecting ends was solved by H. Schulz [380] by means of a method introduced by M. Gaudin [155] for the δ-function Bose gas enclosed in a box and the open spin-$\frac{1}{2}$ Heisenberg chain.

The 'conformal approach' provides information on the large-distance/low-energy behaviour of correlation functions in the Hubbard model for band fillings strictly larger than zero and strictly less than one, for all positive values of U and low temperatures. In the Mott insulating phase at half filling the conformal approach is not applicable. However, in the small-U and scaling limits [324, 490, 491] methods of integrable quantum field theory can be employed to determine dynamical correlation functions at low energies [89, 105, 128, 129, 224] (Chapter 10). Another phase where the conformal approach is not applicable is the so-called gas phase characterized by sufficiently negative chemical potential and hence describing a correlated band insulator. The large-distance asymptotics of finite temperature correlation functions in the gas phase can be obtained exactly by a Bethe ansatz based approach [173, 176] (see Chapter 11). In the strong coupling limit $U \to \infty$ at zero temperature it is possible to obtain information about correlation functions at all energy scales by a combination of analytical and numerical techniques [153, 160, 344, 345, 353–358] for all band fillings.

The Bethe ansatz solution also supplies the coordinate wave functions of eigenstates of the Hamiltonian. An explicit representation for the wave functions was given by F. Woynarovich [481]. Only much later [122, 125] it was proven that the corresponding eigenstates are highest weight states with respect to the SO(4) symmetry [197, 497] of the Hamiltonian (Chapter 3). Combining this result with the implications of the string hypothesis it became possible to give a completeness argument in [123] (Chapter 4). Unfortunately the Bethe ansatz wave functions appear to be too complicated to allow for a direct calculation of correlation functions. Even an expression for the norms (Chapter 3) of the eigenstates has only been conjectured [175] and is still awaiting a proof.

In 1986 B. S. Shastry opened up a new way for studying the Hubbard model by placing it into the framework of the quantum inverse scattering method. Using a Jordan-Wigner transformation he mapped the Hubbard model to a spin model and then demonstrated that the resulting spin Hamiltonian commutes with the transfer matrix of a related covering vertex model [392]. In [391] Shastry first obtained the R-matrix of the spin model, thus embedding it into the general classification of ‘integrable models’ (see Chapter 12). Alternative derivations were obtained in [393] and in [348, 349, 475]. The latter references also include a formulation in terms of fermions that applies more directly to the Hubbard model.

It then took about ten years before Shastry’s construction was really utilized. In [401] it was shown that Shastry’s R-matrix satisfies the Yang-Baxter equation. An algebraic Bethe ansatz for the Hubbard model was constructed in [320, 371] and expressions for the eigenvalues of the transfer matrix of the two-dimensional statistical covering model were obtained (see also [499]). This result was of crucial importance for the quantum transfer matrix approach to the thermodynamics [232] of the Hubbard model (Chapter 13). This approach allows for a drastically simplified description of the thermodynamics in terms of the solution of a finite set of nonlinear integral equations, rather than the infinite set originally obtained by Takahashi in 1972 [435]. Within the quantum transfer matrix approach thermodynamic quantities can be calculated numerically with a very high precision. The approach can be extended to the calculation of correlation lengths at finite temperature [459, 465].

Another important algebraic result, which was unrelated to Shastry’s work at first sight, was the discovery of a quantum group symmetry of the Hubbard model on the infinite line (see Chapter 14): the Hamiltonian is invariant under the direct sum of two Y(sl(2)) Yangians [462] (see also [172]). The relation of these Yangians to Shastry’s R-matrix and the implications of one of these Yangians for the structure of the bare excitations (Chapter 15) were clarified in [335, 336].

1.2.3 The one-dimensional Hubbard model and experiments

The one-dimensional Hubbard model has been of immense conceptual value in facilitating the interpretation of experiments on quasi one-dimensional materials. Although it is not strictly a perfect model for any existing material, many of its qualitative features seem to be realized in nature. At present there is a sizeable list of materials, for which the electronic degrees of freedom are believed to be described by ‘Hubbard-like’ Hamiltonians. Examples