THE GEOMETRY OF MODULI SPACES OF SHEAVES

Second Edition

Now back in print, this highly regarded book has been updated to reflect recent advances in the theory of semistable coherent sheaves and their moduli spaces, which include moduli spaces in positive characteristic, moduli spaces of principal bundles and of complexes, Hilbert schemes of points on surfaces, derived categories of coherent sheaves, and moduli spaces of sheaves on Calabi-Yau threefolds. The authors review changes in the field since the publication of the original edition in 1997 and point the reader towards further literature. References have been brought up to date and errors removed.

Developed from the authors' lectures, this book is ideal as a text for graduate students as well as a valuable resource for any mathematician with a background in algebraic geometry who wants to learn more about Grothendieck's approach.

'This book fills a great need: it is almost the only place the foundations of the moduli theory of sheaves on algebraic varieties appears in any kind of expository form. The material is of basic importance to many further developments: Donaldson-Thomas theory, mirror symmetry, and the study of derived categories.'
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'This is a wonderful book; it's about time it was available again. It is the definitive reference for the important topics of vector bundles, coherent sheaves, moduli spaces and geometric invariant theory; perfect as both an introduction to these subjects for beginners, and as a reference book for experts. Thorough but concise, well written and accurate, it is already a minor modern classic. The new edition brings the presentation up to date with discussions of more recent developments in the area.'
Richard Thomas, Imperial College London

THE GEOMETRY OF MODULI SPACES OF SHEAVES

Second Edition

DANIEL HUYBRECHTS AND MANFRED LEHN

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Preface to the second edition

The first edition of this book has been out of print for some years now. But as the book still seems a useful source for the main techniques, results and open problems in this area and as it has been appreciated by newcomers wanting to learn the material from scratch, there have been frequent requests for a new edition.

Since the first appearance of the book, the field has branched out in various directions: moduli spaces in positive characteristic, moduli spaces of principal bundles and of complexes, Hilbert schemes of points on surfaces, derived categories of coherent sheaves, moduli spaces of sheaves on Calabi-Yau threefolds, and many, many others. In the new appendices we make comments on some of these interesting new research directions without aiming at completeness nor going into the technical details. The main text has been left unchanged as far as possible. We used however the opportunity to correct a number of mistakes known to us and tried to improve the presentation at certain places.

We are aware of a number of excellent recent textbooks that are closely related to the topics treated here. Friedman's book [322] is a good point to start for anybody who wants to learn about vector bundles on surfaces. It combines an introduction to the theory of algebraic surfaces with a study of vector bundles. Moduli spaces of vector bundles on curves are constructed and discussed in Le Potier's book [367]. The monograph of Schmitt [429] gives a very detailed account of GIT phenomena encountered in the construction of moduli spaces of sheaves. Two further new textbooks on GIT by Dolgachev and Mukai are highly recommended, [312] and [394].

Many people have indicated inaccuracies and mistakes in the first edition and have made valuable suggestions on how to improve the text. We would like to thank in particular Holger Brenner, Alastair King, Adrian Langer, and Alexander Schmitt.

We wish to thank Cambridge University Press for convincing us that a new edition was a good idea and for letting us decide to what extent we wanted to modify and augment the previous version.

Preface to the first edition

The topic of this book is the theory of semistable coherent sheaves on a smooth algebraic surface and of moduli spaces of such sheaves. The content ranges from the definition of a semistable sheaf and its basic properties over the construction of moduli spaces to the birational geometry of these moduli spaces. The book is intended for readers with some background in Algebraic Geometry, as for example provided by Hartshorne's textbook [98].

There are at least three good reasons to study moduli spaces of sheaves on surfaces. Firstly, they provide examples of higher dimensional algebraic varieties with a rich and interesting geometry. In fact, in some regions in the classification of higher dimensional varieties the only known examples are moduli spaces of sheaves on a surface. The study of moduli spaces therefore sheds light on some aspects of higher dimensional algebraic geometry. Secondly, moduli spaces are varieties naturally attached to any surface. The understanding of their properties gives answers to problems concerning the geometry of the surface, e.g. Chow group, linear systems, etc. From the mid-eighties till the mid-nineties most of the work on moduli spaces of sheaves on a surface was motivated by Donaldson's ground breaking results on the relation between certain intersection numbers on the moduli spaces and the differentiable structure of the four-manifold underlying the surface. Although the interest in this relation has subsided since the introduction of the extremely powerful Seiberg-Witten invariants in 1994, Donaldson's results linger as a third major motivation in the background; they throw a bridge from algebraic geometry to gauge theory and differential geometry.

Part I of this book gives an introduction to the general theory of semistable sheaves on varieties of arbitrary dimension. We tried to keep this part to a large extent self-contained. In Part II, which deals almost exclusively with sheaves on algebraic surfaces, we occasionally sketch or even omit proofs. This area of research is still developing and we feel that some of the results are not yet in their final form.

Some topics are only touched upon. Many interesting results are missing, e.g. the Fourier-Mukai transformation, Picard groups of moduli spaces, bundles on the projective plane (or more generally on projective spaces, see [230]), computation of Donaldson polynomials on algebraic surfaces, gauge theoretical aspects of moduli spaces (see the book of Friedman and Morgan [71]). We also wish to draw the readers' attention to the forthcoming book of R. Friedman [69].

Usually, we give references and sometimes historical remarks in the Comments at the end of each chapter. If not stated otherwise, all results should be attributed to others. We apologize for omissions and inaccuracies that we may have incorporated in presenting their work.

These notes grew out of lectures delivered by the authors at a summer school at Humboldt-Universität zu Berlin in September 1995. Every lecture was centered around one topic. In writing up these notes we tried to maintain this structure. By adding the necessary background to the orally presented material, some chapters have grown out of size and the global structure of the book has become rather non-linear. This has two effects. It should be possible to read some chapters of Part II without going through all the general theory presented in Part I. On the other hand, some results had to be referred to before they were actually introduced.

We wish to thank H. Kurke for the invitation to Berlin and I. Quandt for the organization of the summer school. We are grateful to F. Hirzebruch for his encouragement to publish these notes in the MPI-subseries of the Aspects of Mathematics. We also owe many thanks to S. Bauer and the SFB 343 at Bielefeld, who supported the preparation of the manuscript.

Many people have read portions and preliminary versions of the text. We are grateful for their comments and criticism. In particular, we express our gratitude to: S. Bauer, V. Brinzanescu, R. Brussee, H. Esnault, L. Göttsche, G. Hein, L. Hille, S. Kleiman, A. King, J. Klein, J. Li, S. Müller-Stach, and K. O'Grady.

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Bielefeld, December 1996 Daniel Huybrechts, Manfred Lehn

Introduction

It is one of the deep problems in algebraic geometry to determine which cohomology classes on a projective variety can be realized as Chern classes of vector bundles. In low dimensions the answer is known. On a curve X any class $c_1 \in H^2(X,\mathbb{Z})$ can be realized as the first Chern class of a vector bundle of prescribed rank r. In dimension two the existence of bundles is settled by Schwarzenberger's result, which says that for given cohomology classes $c_1 \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ and $c_2 \in H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ on a complex surface X there exists a vector bundle of prescribed rank ≥ 2 with first and second Chern class c_1 and c_2, respectively.

The next step in the classification of bundles aims at a deeper understanding of the set of all bundles with fixed rank and Chern classes. This naturally leads to the concept of moduli spaces.

The case $r = 1$ is a model for the theory. By means of the exponential sequence, the set $\mathrm{Pic}^{c_1}(X)$ of all line bundles with fixed first Chern class c_1 can be identified, although not canonically for $c_1 \neq 0$, with the abelian variety $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$. Furthermore, over the product $\mathrm{Pic}^{c_1}(X) \times X$ there exists a 'universal line bundle' with the property that its restriction to $[L] \times X$ is isomorphic to the line bundle L on X. The following features are noteworthy here: Firstly, the set of all line bundles with fixed Chern class carries a natural scheme structure, such that there exists a universal line bundle over the product with X. This is roughly what is called a moduli space. Secondly, if c_1 is in the Neron-Severi group $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$, the moduli space is a nonempty projective scheme. Thirdly, the moduli space is irreducible and smooth. And, last but not least, the moduli space has a distinguished geometric structure: it is an abelian variety. This book is devoted to the analogous questions for bundles of rank greater than one. Although none of these features generalizes literally to the higher rank situation, they serve as a guideline for the investigation of the intricate structures encountered there.

For $r > 1$ one has to restrict oneself to semistable bundles in order to get a separated finite type scheme structure for the moduli space. Pursuing the natural desire to work with complete spaces, one compactifies moduli spaces of bundles by adding semistable non-locally free sheaves. The existence of semistable sheaves on a surface, i.e. the non-emptiness of the moduli spaces, can be ensured for large c_2 while r and c_1 are fixed. Under the same assumptions, the moduli spaces turn out to be irreducible. Moduli spaces of sheaves of rank ≥ 2 on a surface are not smooth, unless we consider sheaves with special invariants on special surfaces. Nevertheless, something is known about the type of singularities they can attain. Concerning the geometry of moduli spaces of sheaves of higher rank, there are two guiding principles for the investigation. Firstly, the geometric structure of sheaves of rank $r > 1$ reveals itself only for large second Chern number c_2 while c_1 stays fixed. In other words, only high dimensional moduli spaces display the properties one expects them to have. Secondly, contrary to the case $r = 1$, $c_2 = 0$, where $\mathrm{Pic}^{c_1}(X)$ is always an abelian variety no matter whether X is ruled, abelian, or of general type, moduli spaces of sheaves of higher rank are expected to inherit geometric properties from the underlying surface. In particular, the position of the surface in the Enriques classification is of uttermost importance for the geometry of the moduli spaces of sheaves on it. Much can be said about the geometry, but at least as much has yet to be explored. The variety of geometric structures exposed by moduli spaces, which in general are far from being 'just' abelian, makes the subject highly attractive to algebraic geometers.

Let us now briefly describe the contents of each single chapter of this book. We start out in Chapter 1 by providing the basic concepts in the field. Stability, as it was first introduced for bundles on curves by Mumford and later generalized to sheaves on higher dimensional varieties by Takemoto, Gieseker, Maruyama, and Simpson, is the topic of Section 1.2. This notion is natural from an algebraic as well as from a gauge theoretical point of view, for there is a deep relation between stability of bundles and existence of Hermite-Einstein metrics. This relation, known as the Kobayashi-Hitchin correspondence, was established by the work of Narasimhan-Seshadri, Donaldson and Uhlenbeck-Yau. We will elaborate on the algebraic aspects of stability, but refer to Kobayashi's book [127] for the analytic side (see also [157]). Vector bundles are best understood on the projective line where they always split into the direct sum of line bundles due to a result usually attributed to Grothendieck (1.3.1). In the general situation, this splitting is replaced by the Harder-Narasimhan filtration, a filtration with semistable factors (Section 1.3). If the sheaf is already semistable, then the Jordan-Hölder filtration filters it further, so that the factors become stable. Following Seshadri, the associated graded object is used to define S-equivalence of semistable sheaves (Section 1.5). Stability in higher dimensions can be introduced in various ways, all generalizing Mumford's original concept. In Section 1.6 we provide a framework to compare the different possibilities. The Mumford-Castelnuovo regularity and Kleiman's boundedness results, which are stated without proof in Section 1.7, are fundamental for the construction of the moduli space. They are needed to ensure that the set of semistable sheaves is small enough to be parametrized by a scheme of finite type. Another important ingredient is Grothendieck's Lemma (1.7.9) on the boundedness of subsheaves.

Moduli spaces are not just sets of objects; they can be endowed with a scheme structure. The notion of families of sheaves gives a precise meaning to the intuition of what this structure should be. Chapter 2 is devoted to some aspects related to families of sheaves. In Section 2.1 we first construct the flattening stratification for any sheaf and then consider flat families of sheaves and some of their properties. The Grothendieck Quot-scheme, one of the fundamental objects in modern algebraic geometry, together with its local description will be discussed in Section 2.2 and Appendix 2.A. In this context we also recall the notion of corepresentable functors which will be important for the definition of moduli spaces as well. As a consequence of the existence of the Quot-scheme, a relative version of the Harder-Narasimhan filtration is constructed. This and the openness of stability, due to Maruyama, will be presented in Section 2.3. In Appendix 2.A we introduce flag-schemes, a generalization of the Quot-scheme, and sketch some aspects of the deformation theory of sheaves, quotient sheaves and, more general, flags of sheaves. In Appendix 2.B we present a result of Langton showing that the moduli space of semistable sheaves is, a priori, complete.

Chapter 3 establishes the boundedness of the set of semistable sheaves. The main tool here is a result known as the Grauert-Mülich Theorem. Barth, Spindler, Maruyama, Hirschowitz, Forster, and Schneider have contributed to it in its present form. A complete proof is given in Section 3.1. At first sight this result looks rather technical, but it turns out to be powerful in controlling the behaviour of stability under basic operations like tensor products or restrictions to hypersurfaces. We explain results of Gieseker, Maruyama and Takemoto related to tensor products and pull-backs under finite morphisms in Section 3.2. In the proof of boundedness (Section 3.3), we essentially follow arguments of Simpson and Le Potier. The theory would not be complete without mentioning the famous Bogomolov Inequality. We reproduce its by now standard proof in Section 3.4 and give an alternative one later (Section 7.3). The Appendix to Chapter 3 uses the aforementioned boundedness results to prove a technical proposition due to O'Grady which comes in handy in Chapter 9.

The actual construction of the moduli space takes up all of Chapter 4. The first construction, due to Gieseker and Maruyama, differs from the one found by Simpson some ten years later in the choice of a projective embedding of the Quot-scheme. We present Simpson's approach (Sections 4.3 and 4.4) as well as a sketch of the original construction (Appendix 4.A). Both will be needed later. We hope that Section 4.2, where we recall some results concerning group actions and quotients, makes the construction accessible even for the reader not familiar with the full machinery of Geometric Invariant Theory. In Section 4.5 deformation theory is used to obtain an infinitesimal description of the moduli space, including bounds for its dimension and a formula for the expected dimension in the surface case. In particular, we prove the smoothness of the Hilbert scheme of zero-dimensional subschemes of a smooth projective surface, which is originally due to Fogarty. In contrast to the rank one case, a universal sheaf on the product of the moduli space and the variety does not always exist. Conditions for the existence of a (quasi)-universal family are discussed in Section 4.6. In Appendix 4.B moduli spaces of sheaves with an additional structure, e.g. a global section, are discussed. As an application we construct a 'quasi-universal family' over a projective birational model of the moduli space of semi-stable sheaves. This will be useful for later arguments. The dependence of stability on the fixed ample line bundle on the variety was neglected for many years. Only in connection with the Donaldson invariants was its significance recognized. Friedman and Qin studied the question from various angles and revealed interesting phenomena. We only touch upon this in Appendix 4.C, where it is shown that for two fixed polarizations on a surface the corresponding moduli spaces are birational for large second Chern number. Other aspects concerning fibred surfaces will be discussed in Section 5.3.

From Chapter 5 on we mainly focus on sheaves on surfaces. Chapter 5 deals with the existence of stable bundles on surfaces. The main techniques are Serre's construction (Section 5.1) and Maruyama's elementary transformations (Section 5.2). With these techniques at hand, one produces stable bundles with prescribed invariants like rank, determinant, Chern classes, Albanese classes, etc. Sometimes, on special surfaces, the same methods can in fact be used to describe the geometry of the moduli spaces. Bundles on elliptic surfaces were quite intensively studied by Friedman. Only a faint shadow of his results can be found in Sections 5.3, where we treat fibred surfaces in some generality and two examples for K3 surface.

We continue to consider special surfaces in Chapter 6. Mukai's beautiful results concerning two-dimensional moduli spaces on K3 surfaces are presented in Section 6.1. Some of the results, due to Beauville, Göttsche-Huybrechts, O'Grady, concerning higher dimensional moduli spaces will be mentioned in Section 6.2. In the course of this chapter we occasionally make use of the irreducibility of the Quot-scheme of all zero-dimensional quotients of a locally free sheaf on a surface. This is a result originally due to Li and Li-Gieseker. We present a short algebraic proof due to Ellingsrud and Lehn in Appendix 6.A.

As a sequel to the Grauert-Mülich theorem we discuss other restriction theorems in Chapter 7. Flenner's theorem (Section 7.1) is an essential improvement of the former and allows one to predict the μ-semistability of the restriction of a μ-semistable sheaf to hyperplane sections. The techniques of Mehta-Ramanathan (Section 7.2) are completely different and allow one to treat the μ-stable case as well. Bogomolov exploited his inequality to prove the rather surprising result that the restriction of a μ-stable vector bundle on a surface to any curve of high degree is stable (Section 7.3).

In Chapter 8 we strive for an understanding of line bundles on moduli spaces. Line bundles of geometric significance can be constructed using the technique of determinant bundles (Section 8.1). Unfortunately, Li's description of the full Picard group is beyond the scope of these notes, for it uses gauge theory in an essential way. We only state a special case of his result (8.1.6) which can be formulated in our framework. Section 8.2 is devoted to the study of a particular ample line bundle on the moduli space and a comparison between the algebraic and the analytic (Donaldson-Uhlenbeck) compactification of the moduli space of stable bundles. We build upon work of Le Potier and Li. As a result we construct algebraically a moduli space of μ-semistable sheaves on a surface. By work of Li and Huybrechts, the canonical class of the moduli space can be determined for a large class of surfaces (Section 8.3).

Chapter 9 is almost entirely a presentation of O'Grady's work on the irreducibility and generic smoothness of moduli spaces. Similar results were obtained by Gieseker and Li. Their techniques are completely different and are based on a detailed study of bundles on ruled surfaces. The main result roughly says that for large second Chern number the moduli space of semistable sheaves is irreducible and the bad locus of sheaves, which are not μ-stable or which correspond to singular points in the moduli space, has arbitrary high codimension.

In Chapter 10 we show how one constructs holomorphic one- and two-forms on the moduli space starting with such forms on the surface. This reflects rather nicely the general philosophy that moduli spaces inherit properties from the underlying surface. We provide the necessary background like Atiyah class, trace map, cup product, Kodaira-Spencer map, etc., in Section 10.1. In Section 10.2 we describe the tangent bundle of the smooth part of the moduli space in terms of a universal family. In fact, this result has been used already in earlier chapters. The actual construction of the forms is given in Section 10.3 where we also prove their closedness. The most famous result concerning forms on the moduli space is Mukai's theorem on the existence of a non-degenerate symplectic structure on the moduli space of stable sheaves on K3 surfaces (Section 10.4). O'Grady pursued this question for surfaces of general type.

Chapter 11 combines the results of Chapter 8 and 10 and shows that moduli spaces of semistable sheaves on surfaces of general type are of general type as well. We start with a proof of this result for the case of rank one sheaves, i.e. the Hilbert scheme. Our presentation of the higher rank case deviates slightly from Li's original proof. Other results on the birational type of moduli spaces are listed in Section 11.2. We conclude this chapter with two rather general examples where the birational type of moduli spaces of sheaves on (certain) K3 surfaces can be determined.