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Excerpt
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Part I

Style and Presentation

1
Solving Problems

What do we mean by 'problem solving', and what benefits should we expect from solving problems? Of course, by solving a given problem we obtain an answer, from which we may be able to draw a conclusion. There are practical problems which arise in industry where it is important, perhaps from financial considerations, to find an answer, and there is only a secondary interest in how the answer has been obtained. In such cases, the answer is the main objective. While I hope that the benefits that accrue from studying this text will improve the reader's skills in solving such problems, we shall focus here on the *process* of solving problems, rather than on the answers. In fact, the answers will be of little interest to us, except in that they illustrate a method, or suggest further investigations.

The primary object of our study is the problem itself, and its main roles are to show us how mathematics can be applied in a variety of ways, to provide motivation for us to learn more mathematics, and to see and experience how simple cases lead to a greater understanding, and hence to further problems, generalisations, and so on. Solving problems in this sense is like a journey of exploration; we must constantly pay attention to the local details, but all the time be aware of how these details fit into a much larger, unknown, picture. It has been said that we do not understand a piece of mathematics unless we can generalise it, and a generalisation usually calls for different ideas. Thus we should see our attempt to solve a particular problem as a continuously evolving account of a wider problem.

Many educationalists favour this experimental approach, believing that one can only learn mathematics by doing mathematics oneself.

This has much to recommend it but, by itself, it cannot be enough. There is no doubt that to succeed in mathematics one needs a vast supply of mathematical knowledge, and one cannot be expected to provide this entirely by one's own efforts. At some stage we must learn from others (Newton's phrase 'standing on the shoulders of giants' springs to mind here), so what is the best way to learn more mathematics? This text is an attempt to show how problems can motivate this learning.

How should we proceed when faced with a non-standard problem? In general, we will not be asked to show that a certain formula is true; we will simply be asked to "Investigate the following situation: . . . ". Suppose that we have never seen any problem like this before; how do we begin? First, if we are to use mathematics, we need to formulate the problem in mathematical terms and sometimes this can be quite difficult. It is, however, an essential step and it is often the key to a solution. Sometimes the problem will not be properly defined; for example, if we need to describe a pattern of coloured discs, are we to regard two patterns as being the same if one is a rotation of the other? What if one is a reflection of the other? There is a decision to be taken here, and it must be emphasised that the responsibility of taking such decisions *lies with the problem-solver*. In truth, each decision will represent a different problem. Thus, when carefully formulated, the original problem is often transformed into several different problems.

Next, we should try to specialise the problem, or try simple cases; indeed, do anything that gives us more insight into the situation, and which enables us to 'tune in' to the problem. It is reasonable to suppose that the simpler cases will be easier to solve than the general problem, but there is a danger here for it is often possible to solve a simpler case by a method which will not be applicable in the more general case. As long as we are aware of this, no harm (but some good) will have been done. The results of simple cases may lead us to conjectures, and these should be checked wherever possible by use of a computer, geometry, graphics, sketches, rough calculations, and so on. If a conjecture turns out to be false, we should not be disappointed; we must find the error in our thinking and then more progress will have been made. Sometimes, the tools (matrices, groups, calculus, . . .) needed to solve a problem will be clear, but they may need to be refined. For example, we may suspect that we need something about numerical congruences, but that our knowledge is not enough. In this case an hour or so browsing though appropriate texts in the library might be helpful. If, at any stage, we realise that our problem, or proposed solution, has anything to do with some other part of mathematics, we should take time off to study this. Solving problems in the sense described in this text takes time, and the reader should not expect to solve the problem in one sitting. Experience shows that our thoughts about a problem will evolve with time, so we must give the problem the time it needs and deserves.

Suppose that at some stage we seem to have reached a dead end, and all our efforts have failed. Maybe this is because we are looking at the problem in the wrong way. One of the most attractive and powerful features of mathematics is the way in which the same thing can be said in many seemingly different, but actually equivalent, languages. It should not surprise the reader to learn that some of these languages are more easily adapted to solve a given problem than others, so perhaps the real issue is to find which of the languages is the best one to use. For instance, a problem involving integers might best be expressed in terms of congruences, or in terms of prime numbers, or in terms of binary numbers, or in terms of a group in modular arithmetic. Finding out which (if any) of these is the best to use is part of the skill in problem solving, so we should constantly think about the possibility that another language might be better suited to the problem under consideration. Very often, the process of generalisation requires us to change from a description in one language to another.

It is important that from time to time we write a carefully crafted account of what we have achieved, what we hope to achieve, and what we have tried, but failed, to do. This very exercise of putting our thoughts on paper in a coherent way will (provided that it is treated seriously) often give a further stimulus, or provide new ideas, for it requires us to think, and organise our thoughts, about our problem. We should take every opportunity to discuss the problem, and our ideas for a solution, with friends, colleagues and others. Ultimately, mathematics is about *sharing* ideas, for only in this way can we, as a group, extend our collective knowledge of mathematics in an efficient way.

Let us now suppose that we have solved the problem; have we finished? Surely not! First, have we really understood our solution, or do we have some irrelevant, or redundant, mathematics in it? If so, then we should rewrite the solution without it. Eventually, when we have removed this unneccesary material, we can concentrate on the heart of our solution. What have we used, and what have we assumed? Does this method solve a more general problem? If we change some of the parameters in the problem do we still have a solution? If so, we should acknowledge this; if not, then we have another problem which we should also try to solve. The reader should realise that once a problem is solved, there still remains the task of *reviewing, polishing, and rewriting the solution until it finally appears as an elegant piece of carefully argued mathematics*. One should not include one's first (probably wayward, sometimes incorrect, and often irrelevant) attempts in the 'final' write-up of the problem!

Finally, *a problem is never finished*, and it is for the reader to decide at which point it will be more useful to move to another problem, or more fun to do something else, or just relax.

2
Writing Mathematics

There are many circumstances in which we may be asked to write some mathematics or, more generally, a scientific report. It is not easy to write a good report, and it takes a lot of time to learn, but it is a skill that most people can learn. You may find that writing a report is more difficult than giving a lecture. When we give a lecture we can assess how the audience is reacting, and we can say more, or less, in response to this. We cannot do this when writing a report. Here we offer some general suggestions and hints on how to write a good report. We do not specify the nature of the report, and, for obvious reasons, we direct much of our attention towards the writing of mathematics (which we continue to refer to as a 'report'). This is *not* a set of golden rules to follow; rather, it is a list of points to think about, and consider, before, and during, the writing of the report. It should help us decide what to include and what to omit.

Before we consider the typical structure of a report, you should consider the following important questions.

- *Why are you writing a report*?
- *Who will read the report, and what are they looking for*?

In order to write a good report we must be clear about the purpose of our report: *what do we want to achieve, and what is our message*? Keep the purpose of the report in mind at all times; we should have an identifiable reason for including each item in the report.

We should *think carefully about the level of knowledge of our intended readers, and what they want to achieve by reading the report*, and write accordingly. It is very easy to forget who we are writing for, and at each stage of the writing we should ask what will the reader think of this? Also, be consistent; for example, do not solve quadratic equations while discussing an advanced topic in topology! At any given level some things are obvious and do not need to be said, while other things may need considerable explanation; in each case consider which of these is applicable.

Let us now consider the structure of a typical report. Most reports will have something like the following sections:

1. Title
2. Abstract
3. Contents
4. Introduction
5. The main body of the report
6. Conclusion
7. Acknowledgments
8. References

The first four sections should be designed to lead the reader gently into the main body of your report. We should include a statement of what we want to achieve, and a summary of our conclusions. These sections should give the readers a sense of direction; readers do not want to go on a journey with no idea of where they are going. A brief description of these sections now follows.

Title This should be informative (not, for example 'On a theorem in analysis').

Abstract The sole purpose of the abstract is to enable potential readers to decide whether or not they want to read the report. It should be entirely self-contained, and it should not be necessary to consult other works to understand the abstract. Usually, it will contain a brief non–technical description (without symbols) of the contents of the report.

Contents This tells the reader how the report is organised; it is a road map of the report. Choose the titles of each chapter/section carefully; they should give the reader a clear indication of what each chapter/section contains.

Introduction This should provide an easy entry point for the reader. The introduction usually includes some background material (sometimes the motivation for the report), a statement about what we are assuming that the reader already knows, and what we will achieve in the report. It is often an elaboration of the abstract, but these two sections have a different purpose. The abstract enables the reader to decide whether or not they want to read further; once they have decided to continue reading, the introduction is taken as an agreed starting point for the rest of the report.

The main body of the report This section should contain the bulk of the report and, broadly speaking, should follow the lines suggested in the Introduction. This part of the report should contain a discussion of the work that the report depends on, detailed arguments and analysis, and so on, leading to the conclusions. It is a good idea to insert figures last, because creating and changing figures can take a lot of time.

Conclusion This is a summary of what we have achieved, and we should state whether we have achieved the objectives as set out earlier in the report. If appropriate, some open problems or suggestions for further work can be included here.

Acknowledgments Here we acknowledge any assistance or joint work in the report.

References The references are numbered, and are usually given in alphabetical order (in mathematical writings), although there are other accepted conventions (for example, in the order in which they appear in the report). References in the body of the report are usually indicated by, for example, [2]. We are entitled to refer to, and use, results that are already in the literature but we must refer to them explicitly and list the reference in this section.

We end this section with some points to consider when writing the report. First, the structure of the report should be coherent, and it should make it easy for the reader to follow the arguments. Before we start writing we should sketch a brief outline of the structure. We should think about what to include in each section, and we should ensure that we have a reason for including each item. Also, we should think about the length of each section. It is a good idea to write a rough contents page and allocate page numbers. It doesn't matter at all if this allocation changes as we write, but it will act as a guide while we are writing the report. Moreover, if there is a length restriction, this will help us conform to it.

Pay attention to details of the structure of the report. A new section indicates a strong division of the material; a new paragraph indicates a much weaker division. Within a section, paragraphs should be loosely related. As we read/write the report we should ask ourselves whether there are too many different ideas in a paragraph. Generally, a sentence should express only one idea. Be aware of the number of ideas in a sentence or paragraph. If we introduce too many ideas at once, the reader will find it difficult to understand what is written. Also, do not give the reader information before they need it for they will have probably forgotten it by the time we refer to it.

In some ways writing a report can be more difficult than giving a lecture. When giving a lecture we can immediately assess whether or not the audience understands what we are saying, and we can then choose to say more or less as appropriate; indeed, we can even repeat ourself (several times if necessary), which we should not do in a report. Throughout the report we must be careful to say clearly and exactly what we mean so that the readers do not require further, or a repeated, explanation. We should expect to rewrite the report many times before we get it right.

There are several ways to write the main body of the report. Some people begin writing the report with the main body because, in their view, it is easier to write the Introduction and Abstract after this. Other people start with the first line of the report and then, after a while, go back to the beginning and re-write all that has been written so far (repeating this process many times). Yet another way, if we think of the report as a human body, is to build the skeleton first and then add the flesh to the report. If we feel that a section is getting too long and complicated it is a good idea to inform the reader where we are in the grand scheme of things. It is the author's responsibility to tell the reader where they are at present, and where they are going next. We should