1 Radiation and initial-value problems for the wave equation

1.1 The radiation problem

We consider the radiation of waves from a real-valued space- and time-varying source $q(\mathbf{r}, t)$ embedded in an infinite, homogeneous, isotropic and non-dispersive and non-attenuating medium such as free space. The real-valued radiated wavefield satisfies the inhomogeneous scalar wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] u(\mathbf{r}, t) = q(\mathbf{r}, t) \tag{1.1}$$

throughout all of space and for all time, where c is the constant velocity of wave propagation in the background medium. We will assume that the source q is compactly supported in the space-time region $\{\mathcal{S}_0 | \mathbf{r} \in \tau_0,\ t \in [0, T_0]\}$, where τ_0 is its spatial volume and $[0, T_0]$ the interval of time over which the source is turned on. We also assume that the source possesses finite energy (is square-integrable in $\mathcal{S}_0$)

$$\mathcal{E}_q = \int_0^{T_0} dt \int_{\tau_0} d^3r |q(\mathbf{r}, t)|^2 < \infty, \tag{1.2}$$

although we will sometimes have to enlarge the class of sources to include Dirac delta functions, which are not square-integrable, but these cases are special and will be dealt with on an individual basis as required.

The reader may wonder why we have assumed that the source radiates only over a finite time period $[0, T_0]$ as opposed to being allowed to radiate over the semi-infinite interval $[0, \infty)$. The main reason is that it simplifies the mathematics without being a real restriction on the theory and results that we will obtain. In particular, although the time interval over which the source radiates is finite, it can be arbitrarily large so that this source model can apply to any real source to arbitrary accuracy. Moreover, most of our results will be valid in the limit $T_0 \to \infty$ so that the assumption places little or no restriction on our theoretical development.

The solution to the inhomogeneous wave equation Eq. (1.1) is not unique. In particular, it is clear that we can add to u any solution to the *homogeneous wave equation*

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] \delta u(\mathbf{r}, t) = 0 \tag{1.3}$$

and still obtain a solution to Eq. (1.1). In order to obtain a unique solution it is necessary to specify *initial conditions* in the form of the *Cauchy conditions*

$$u(\mathbf{r},t)|_{t=0} = u_0(\mathbf{r}), \qquad \frac{\partial}{\partial t}u(\mathbf{r},t)|_{t=0} = u_0'(\mathbf{r}), \tag{1.4}$$

where u_0 and u_0' are arbitrary (real-valued) functions of position $\mathbf{r}$. We will show below that the inhomogeneous wave equation Eq. (1.1) together with Cauchy conditions at $t = 0$ suffice to uniquely determine the field u. The appropriate initial conditions required of the physically meaningful solution to Eq. (1.1) are derived from the requirement of *causality*; i.e., we seek the *particular solution* to the wave equation $u_+(\mathbf{r},t)$ that is causally related to the source; i.e., that vanishes prior to the turn-on time of the source ($t = 0$). The required causality of the field u_+ is equivalent to requiring that this field satisfy *homogeneous Cauchy conditions* at $t = 0$; i.e.,

$$u_+(\mathbf{r},t)|_{t=0} = 0, \qquad \frac{\partial}{\partial t}u_+(\mathbf{r},t)|_{t=0} = 0,$$

where we have denoted the causal solution to Eq. (1.1) with the subscript $+$. The problem of solving the inhomogeneous wave equation Eq. (1.1) under the condition of causality (or, equivalently, homogeneous Cauchy conditions at $t = 0$) is called the *radiation problem*. The problem of solving the *homogeneous* wave equation Eq. (1.3) subject to arbitrary inhomogeneous Cauchy conditions at $t = 0$ is called the *initial-value problem*. We will treat both problems in this chapter.

1.1.1 Fourier integral representations

We will make frequent use of Fourier integral representations of space and time dependent functions and fields throughout the book. We will assume throughout that any function (or field) $f(\mathbf{r},t)$ possesses a temporal Fourier transform defined by

$$F(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, f(\mathbf{r},t)e^{i\omega t}, \tag{1.5a}$$

and that the temporal transform can be further transformed via a spatial Fourier transform of the form

$$\tilde{F}(\mathbf{K},\omega) = \int d^3r\, F(\mathbf{r},\omega)e^{-i\mathbf{K}\cdot\mathbf{r}}, \tag{1.5b}$$

where the integration in Eq. (1.5b) is carried out over all of R^3. We further assume that each of the transforms can be inverted to yield Fourier integral representations given by

$$f(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\mathbf{r},\omega)e^{-i\omega t}, \tag{1.6a}$$

$$F(\mathbf{r},\omega) = \frac{1}{(2\pi)^3}\int d^3K\, \tilde{F}(\mathbf{K},\omega)e^{i\mathbf{K}\cdot\mathbf{r}}. \tag{1.6b}$$

We can, of course, combine the temporal and spatial Fourier integral representations into a single *space-time* representation of the form

$$f(\mathbf{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \int d^3K\, \tilde{F}(\mathbf{K},\omega)e^{i(\mathbf{K}\cdot\mathbf{r}-\omega t)}, \tag{1.7a}$$

where

$$\tilde{F}(\mathbf{K},\omega) = \int_{-\infty}^{\infty} dt \int d^3r\, f(\mathbf{r},t)e^{-i(\mathbf{K}\cdot\mathbf{r}-\omega t)}. \tag{1.7b}$$

As in the presentation given above, we will generally denote functions of space and time (space-time) by lower-case letters and their temporal (time) transforms by upper-case letters. The spatial Fourier transforms of the latter transforms are then represented by an upper-case letter with a tilde on top. Thus, we have the progression

$$f(\mathbf{r},t) \Longleftrightarrow F(\mathbf{r},\omega) \Longleftrightarrow \tilde{F}(\mathbf{K},\omega), \tag{1.8}$$

where the double arrow $\Longleftrightarrow$ denotes the Fourier-transform operation. We will explicitly display the limits on one-dimensional integrals as in the temporal and inverse temporal transforms given above but will not explicitly display the limits on multi-dimensional integrals unless their integration domains are finite.

The classical theory of the Fourier integral requires that the functions $f(\mathbf{r},t)$, $F(\mathbf{r},\omega)$ and $\tilde{F}(\mathbf{K},\omega)$ are all absolutely integrable and that the integrals be interpreted as Lebesgue integrals for the above set of equations to hold. If, in addition, the functions decay sufficiently fast at infinity they will possess the important property that *multiplication by the frequency variable in the frequency domain corresponds to differentiation in the time or space domain*. The modern theory of the Fourier integral is based on distribution theory (the theory of generalized functions) and avoids all of the analysis and issues of the classical theory as well as enlarging the class of functions that can be transformed to include discontinuous and non-differentiable (generalized) functions. Within the context of distribution theory any generalized function $f(\mathbf{r},t)$ will possess transforms and inverse transforms as given above and partial derivatives that are related to their transforms via the equations

$$\frac{\partial^n}{\partial t^n}f(\mathbf{r},t) \Longleftrightarrow (-i\omega)^n F(\mathbf{r},\omega), \qquad \frac{\partial^n}{\partial x^n}F(\mathbf{r},\omega) \Longleftrightarrow (iK_x)^n \tilde{F}(\mathbf{K},\omega), \tag{1.9}$$

where n is any positive integer, x is any of the Cartesian components of $\mathbf{r}$, and the double arrow $\Longleftrightarrow$ denotes the Fourier-transform operation.

Important examples of generalized functions are the Dirac delta functions which are defined according to the equations[1]

[1] We will not employ different symbols for one-, two- and three-dimensional delta functions unless their dimensionality is not clear from their argument.

$$\phi(0) = \int_{-\infty}^{\infty} dt\, \delta(t)\phi(t), \qquad \chi(\mathbf{0}) = \int d^3r\, \delta(\mathbf{r})\chi(\mathbf{r}), \tag{1.10}$$

where $\phi(t)$ and $\chi(\mathbf{r})$ are any well-behaved ordinary functions of t and $\mathbf{r}$, respectively. The Dirac delta functions do not have meaning within the framework of classical function theory and must be interpreted within the framework of distribution theory, where the "integrals" in the above definitions are taken to be inner products defined on a suitable space of "testing functions." Although $\delta(t)$ and $\delta(\mathbf{r})$ are not ordinary functions, they can be formally manipulated and treated as such as long as at the end of a calculation they appear in integrals with ordinary functions that can then be given meaning through Eqs. (1.10). In this connection, we note that the Fourier transforms of the delta functions are given by

$$1 = \int_{-\infty}^{\infty} dt\, \delta(t)e^{i\omega t}, \qquad 1 = \int d^3r\, \delta(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}, \tag{1.11a}$$

which follows from Eqs. (1.10) on taking $\phi(t) = \exp(i\omega t)$ and $\chi(\mathbf{r}) = \exp(-i\mathbf{K}\cdot\mathbf{r})$. The delta functions then admit Fourier-integral representations given by

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}, \qquad \delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3K\, e^{i\mathbf{K}\cdot\mathbf{r}}. \tag{1.11b}$$

We will interpret the Fourier integral throughout this book within the context of distribution theory, which amounts to using the transforms and inverse transforms in a purely formal way without any regard for the properties of the functions being transformed or inverse transformed. In most cases the results we obtain will hold within the classical theory of the transform but will be obtained using much less effort than would be required using the classical theory. In some cases the results cannot be obtained using classical theory but have a perfectly acceptable interpretation within distribution theory as, for example, will be the case when we compute the Green function of the wave equation in the following section. We will not detour into a review of distribution theory but will present certain results from the theory when needed. We refer the interested reader to the books on the subject listed at the end of the chapter.

Example 1.1 As a simple example of the use of the Fourier transform we consider the initial-value problem for the one-dimensional homogeneous wave equation

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]u(z,t) = 0,$$

together with the Cauchy conditions

$$u(z,t)|_{t=0} = u_0(z), \qquad \frac{\partial}{\partial t}u(z,t)|_{t=0} = u_0'(z).$$

It is easy to verify that the general solution is given by

$$u(z,t) = f(z-ct) + g(z+ct), \tag{1.12}$$

where f and g are arbitrary functions that have derivatives up to second order. These functions are uniquely determined from the Cauchy conditions via the equations

$$f(z)+g(z)=u_0(z), \qquad -\frac{\partial}{\partial z}f(z)+\frac{\partial}{\partial z}g(z)=\frac{1}{c}u_0'(z).$$

The above coupled set of equations is easily solved using the Fourier transform. In particular, on Fourier transforming both sides of the above equations we obtain the result

$$\tilde{f}(K)+\tilde{g}(K)=\tilde{u}_0(K), \qquad -iK\tilde{f}(K)+iK\tilde{g}(K)=\frac{1}{c}\tilde{u}_0'(K),$$

where

$$\tilde{f}(K)=\int_{-\infty}^{\infty}dz\,f(z)e^{-iKz}$$

and similarly for the other transformed quantities. We conclude that

$$\tilde{f}(K)=\frac{1}{2}\left[\tilde{u}_0(K)+\frac{i}{cK}\tilde{u}_0'(K)\right],$$
$$\tilde{g}(K)=\frac{1}{2}\left[\tilde{u}_0(K)-\frac{i}{cK}\tilde{u}_0'(K)\right],$$

so that the solution to the Cauchy initial-value problem is given by Eq. (1.12) with

$$f(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dK\,\overbrace{\frac{1}{2}\left[\tilde{u}_0(K)+\frac{i}{cK}\tilde{u}_0'(K)\right]}^{\tilde{f}(K)}e^{iKz},$$
$$g(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dK\,\overbrace{\frac{1}{2}\left[\tilde{u}_0(K)-\frac{i}{cK}\tilde{u}_0'(K)\right]}^{\tilde{g}(K)}e^{iKz}.$$

Example 1.2 In many cases treated here and in later chapters the temporal and/or spatial Fourier transforms are not only ordinary (as opposed to generalized) functions but also analytic functions of the transform variables. As an example, consider the temporal transform $Q(\mathbf{r},\omega)$ of the source $q(\mathbf{r},t)$, which is assumed to be square-integrable and supported in the finite time interval $[0,T_0]$. This transform is an entire analytic function of ω throughout the entire complex-ω plane. To show this we set $f(\mathbf{r},t) = q(\mathbf{r},t)$ in Eq. (1.5a) and expand the exponential in a Taylor series centered at the origin. We then interchange the orders of summation and integration (which is allowable since the integral has finite limits and the series converges uniformly) to obtain the result

$$Q(\mathbf{r},\omega)=\sum_{n=0}^{\infty}A_n\omega^n, \tag{1.13}$$

where $Q(\mathbf{r},\omega)$ is the temporal transform of $q(\mathbf{r},t)$ and

$$A_n=\frac{(i)^n}{n!}\int_0^{T_0}dt\,t^n q(\mathbf{r},t).$$

Since the time-dependent source $q(\mathbf{r},t)$ is square-integrable and compactly supported in $[0,T_0]$, it must be at least piecewise continuous within this interval so that

$$|A_n| \le \frac{\max|q(\mathbf{r},t)|}{n!}\int_0^{T_0} dt\, t^n = \frac{\max|q|T_0^{(n+1)}}{(n+1)!}.$$

It then follows that the Taylor series of $Q(\mathbf{r},\omega)$ given in Eq. (1.13) is term by term smaller than the series

$$G(\omega) = \sum_{n=0}^{\infty} \frac{\max|q|T_0^{(n+1)}}{(n+1)!}\omega^n. \tag{1.14}$$

But the latter series has, by the ratio test, an infinite radius of absolute convergence, i.e.,

$$\frac{\left|\frac{\text{Max}|q|T_0^{(n+2)}}{(n+2)!}\omega^{n+1}\right|}{\left|\frac{\text{Max}|q|T_0^{(n+1)}}{(n+1)!}\omega^{n}\right|} = T_0\frac{|\omega|}{n+2} \to 0, \quad n\to\infty,\ \forall\omega.$$

It then follows by the comparison test that the series Eq. (1.13) also converges absolutely for all ω and, hence, is entire analytic.

1.2 Green functions

> The commonly used expressions "the Green's function" and "a Green's function" represent an atrocity to the English language. I doubt that those who use them ever refer to "a Shakespeare's sonnet." (Rohrlich, 1965)

We define a *Green function* of the wave equation to be any (real) solution to the wave equation Eq. (1.1) for the special case in which the source $q(\mathbf{r},t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, where $\delta(\cdot)$ is the Dirac delta function and $\mathbf{r}'$ and t' are considered to be free parameters that can assume any values in space-time. A Green function to the wave equation thus satisfies the equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] g(\mathbf{r},\mathbf{r}',t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{1.15}$$

It is seen that a Green function $g(\mathbf{r},\mathbf{r}',t,t')$ is simply the field radiated by an impulsive source located at the space-time point $\mathbf{r}',t'$. It follows from the homogeneity of infinite free space and time that any physically meaningful Green function must then be a function only[2] of the difference vector $\mathbf{R} = \mathbf{r}-\mathbf{r}'$ and the time difference[3] $\tau = t-t'$; i.e.,

[2] This follows from the fact that any two space-time points in infinite homogeneous isotropic space-time are indistinguishable; i.e, since there are no physical boundaries or inhomogeneities all space-time points are equivalent so that solutions to the wave equation must be translationally invariant in space-time.

[3] We will also use the Greek symbol τ to denote regions of space throughout the book; e.g., τ_0 as the space support of a source $q(\mathbf{r},t)$ to the wave equation. No confusion should arise, however, since the meaning of the symbol will be clear from the context.

$$g(\mathbf{r}, \mathbf{r}', t, t') = g(\mathbf{r} - \mathbf{r}', t - t') = g(\mathbf{R}, \tau).$$

We can thus replace Eq. (1.15) by the simpler equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right] g(\mathbf{R}, \tau) = \delta(\mathbf{R})\delta(\tau), \tag{1.16}$$

where the Laplacian operator ∇^2 is taken with respect to the components of the $\mathbf{R}$ vector.

A Green function, like any solution to the inhomogeneous wave equation, is not unique. In particular, given any Green function $g(\mathbf{R}, \tau)$ we can obtain a new Green function by adding *any* function $\delta g(\mathbf{R}, \tau)$ that satisfies the homogeneous wave equation Eq. (1.3). The different Green functions obtained in this way will satisfy the same defining equation Eq. (1.16) but different *initial conditions*. As discussed earlier, the choices of initial conditions that result in a unique solution of the inhomogeneous wave equation are known to be the value of the field and its first time derivative at some initial time (the Cauchy conditions). A time-domain Green function $g(\mathbf{R}, \tau)$ to the wave equation in infinite free space is thus uniquely determined by Cauchy conditions at $\tau = 0$.

To compute a Green function we represent it in the space-time Fourier integral representation given in Eqs. (1.7) and make use of Eqs. (1.9). We then find that

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right] g(\mathbf{R}, \tau) \Longleftrightarrow \left[-K^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\mathbf{K}, \omega),$$

where $\Longleftrightarrow$ denotes a four-dimensional Fourier transformation and $K^2 = \mathbf{K}\cdot\mathbf{K}$. The space-time Fourier transform of Eq. (1.16) is then found to be

$$[-K^2 + k^2]\tilde{G}(\mathbf{K}, \omega) = 1,$$

where $k = \omega/c$ is the "wavenumber" of the background medium and we have used the fact that the transforms of the delta functions in Eq. (1.16) are each unity. On using the inverse space-time Fourier transform we then obtain

$$g(\mathbf{R}, \tau) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \int d^3K \, \frac{e^{i(\mathbf{K}\cdot\mathbf{R} - \omega\tau)}}{-K^2 + k^2}. \tag{1.17}$$

1.2.1 Retarded and advanced Green functions

As discussed above, a Green function is not unique and, indeed, the Fourier integral representation Eq. (1.17) does not uniquely define a Green function due to the improper nature of the integral resulting from the poles of the integrand at $k = \omega/c = \pm K$. In order to give meaning to the integral it is necessary to deform the ω contour to avoid these poles, which is possible in a number of ways, with each scheme yielding a different Green function. The Green function of most physical interest is the *causal* Green function, which vanishes for negative time $\tau < 0$ and results in homogeneous Cauchy conditions at $\tau = 0$. The causal Green function, which is referred to as the *retarded Green function* for reasons discussed below, results from deforming the ω integration contour to lie above both poles, leaving the upper half-plane (u.h.p.) free of singularities. If $\tau < 0$ the contour can be closed in the u.h.p., from which it follows from Cauchy's integral theorem that the integral vanishes and the Green function is causal. If $\tau > 0$ we can close the ω integration contour in the lower half-plane (l.h.p.) to obtain

$$g_+(\mathbf{R},\tau) = \frac{c^2}{(2\pi)^4}\int d^3K\, e^{i\mathbf{K}\cdot\mathbf{R}}\int_C d\omega\, \frac{e^{-i\omega\tau}}{\omega^2 - c^2K^2}, \quad \tau > 0, \tag{1.18}$$

where C is the causal contour that lies above both poles and is closed over an infinite semicircle in the l.h.p., and where we have used the subscript + to denote the causal Green function. We can now evaluate the integral using residue calculus to find that

$$g_+(\mathbf{R},\tau) = -\frac{c}{(2\pi)^3}\int d^3K\, e^{i\mathbf{K}\cdot\mathbf{R}}\frac{\sin(cK\tau)}{K}, \quad \tau > 0. \tag{1.19}$$

To finish the calculation we transform to spherical polar integration variables in Eq. (1.19) with the polar axis aligned along the direction of $\mathbf{R}$. We then have that $\mathbf{K}\cdot\mathbf{R} = KR\cos\theta$, with θ the polar angle of $\mathbf{K}$. The integration over the azimuthal angle in Eq. (1.19) can then be performed, and we obtain

$$\begin{aligned} g_+(\mathbf{R},\tau) &= -\frac{c}{(2\pi)^2}\int_0^\infty K\,dK\, \sin(cK\tau)\int_0^\pi e^{iKR\cos\theta}\sin\theta\, d\theta \\ &= -\frac{c}{(2\pi)^2R}\int_{-\infty}^\infty dK\, \sin(cK\tau)\sin(KR), \quad \tau > 0. \end{aligned}$$

The final step is to expand the sine functions using Euler's identity and use the Fourier-integral representation of the delta function given in Eq. (1.11b). We obtain after some minor algebra

$$g_+(\mathbf{R},\tau) = -\frac{1}{4\pi}\frac{\delta(\tau - R/c)}{R}, \tag{1.20a}$$

and $g_+(\mathbf{R},\tau) = 0$ for $\tau < 0$.

As mentioned above, the causal time-domain Green function defined in Eq. (1.20a) is generally known as the *retarded Green function*. The motivation for this nomenclature is that $g_+(\mathbf{r}-\mathbf{r}', t-t')$ represents the field radiated from an impulsive source located at the space-time point $\mathbf{r}'$, t' and this field is not observed at the space point $\mathbf{r}$ until the time $\tau = R/c \Rightarrow t = t' + |\mathbf{r}-\mathbf{r}'|/c$; i.e., the observation time is *retarded* by the distance between the two field points divided by the velocity c of the background medium.

Another time-domain Green function of interest is the acausal or *advanced* Green function $g_-(\mathbf{R},\tau)$. This Green function results from taking the ω contour of integration in the Fourier-integral representation Eq. (1.17) to lie below the two poles. In this case the l.h.p. is free of singularities and the integral vanishes if $\tau > 0$, resulting in the acausal Green function. If $\tau < 0$ we can close the contour in the u.h.p. and, following steps almost identical to those used to compute $g_+(\mathbf{R},\tau)$, we obtain

$$g_-(\mathbf{R},\tau) = -\frac{1}{4\pi}\frac{\delta(\tau + R/c)}{R}, \tag{1.20b}$$

and $g_-(\mathbf{R},\tau) = 0$ for $\tau > 0$. The term *advanced* Green function comes from the property that $g_-(\mathbf{r}-\mathbf{r}', t-t')$ is observed at the space point $\mathbf{r}$ at the time $\tau = -R/c \Rightarrow t = t' - |\mathbf{r}-\mathbf{r}'|/c$; i.e., the observation time is *before* the pulse is emitted: it is *advanced* by the distance between the two field points divided by the velocity c of the background medium. Although the advanced Green function does not arise naturally in the solution of any physical problem, it does play an important role in the class of *inverse problems* treated in later chapters.

1.2.2 Frequency-domain Green functions

Any of the time-domain Green functions $g(\mathbf{R},\tau)$ can be represented according to Eq. (1.6a) in terms of a *frequency-domain Green function* $G(\mathbf{R},\omega)$ with

$$G(\mathbf{R},\omega) = \int_{-\infty}^{\infty} d\tau\, g(\mathbf{R},\tau)e^{i\omega\tau}, \qquad g(\mathbf{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\mathbf{R},\omega)e^{-i\omega\tau}. \tag{1.21}$$

The frequency-domain Green functions are solutions to a partial differential equation, called the reduced wave equation or *Helmholtz equation*, that results from performing a temporal Fourier transform of the wave equation Eq. (1.15) satisfied by the time-domain Green functions $g(\mathbf{R},\tau)$. On making use of Eqs. (1.9) we conclude that

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right] g(\mathbf{R},\tau) \Longleftrightarrow \left[\nabla + \frac{\omega^2}{c^2}\right] G(\mathbf{R},\omega),$$

which then yields the inhomogeneous Helmholtz equation

$$[\nabla^2 + k^2]G(\mathbf{R},\omega) = \delta(\mathbf{R}). \tag{1.22}$$

Like the inhomogeneous wave equation satisfied by the time-domain Green function Eq. (1.16), the inhomogeneous Helmholtz equation Eq. (1.22) does not possess a unique solution until an appropriate *boundary condition* is appended. The requirement of causality in the time domain yields a boundary condition in the frequency domain known as the *Sommerfeld radiation condition* (SRC) (Sommerfeld, 1967) and a Green function denoted by $G_+(\mathbf{R},\omega)$ that is generally referred to as the "outgoing-wave" Green function for reasons to be discussed below. The outgoing-wave Green function can be computed from the Helmholtz equation Eq. (1.22) using a Fourier-based scheme entirely parallel to that used to compute the retarded Green function (cf. Example 1.3 below and our derivation of G_+ presented in the next chapter). Alternatively, it can be obtained by simply taking the temporal Fourier transform of the causal (retarded) Green function found above. Using the latter procedure we obtain

$$G_+(\mathbf{R},\omega) = -\frac{1}{4\pi}\frac{e^{ikR}}{R}, \tag{1.23a}$$

with $k = \omega/c$.

The justification for the use of the name "outgoing-wave" Green function for G_+ is apparent when we examine the Fourier-integral representation of the retarded Green function in terms of G_+:

$$g_+(\mathbf{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left\{-\frac{1}{4\pi}\frac{e^{-i\omega(\tau-R/c)}}{R}\right\}.$$

In particular, we can regard this representation as a superposition of elemental spherical time-harmonic waves

$$u_+(\mathbf{R},\tau) = -\frac{1}{4\pi}\frac{e^{ik(R-c\tau)}}{R},$$

which expand *outward* from the origin $\mathbf{R} = 0$ with increasing time τ. This can be visualized by keeping the phase factor $R - c\tau$ fixed and seeing that increasing τ requires that the distance R must increase also in order for the phase to remain constant. An "incoming wave" on the other hand would be of the form

$$u_-(\mathbf{R},\tau) = -\frac{1}{4\pi}\frac{e^{ik(R+c\tau)}}{R},$$

and would have the property that the surfaces of constant phase contract *inward* toward the origin with increasing τ.

The frequency-domain Green function corresponding to the advanced Green function $g_-(\mathbf{r},\tau)$ is the "incoming-wave" Green function $G_-(\mathbf{R},\omega)$. On taking a temporal Fourier transform of g_- we find that

$$G_-(\mathbf{R},\omega) = -\frac{1}{4\pi}\frac{e^{-ikR}}{R} = G_+^*(\mathbf{R},\omega), \tag{1.23b}$$

where the wavenumber k is assumed to be real-valued. It is easy to verify using an argument similar to that employed above for G_+ that G_- is associated with incoming waves, thus justifying its name as the *incoming-wave* Green function. The Green functions g_- and G_- are associated with the important operations of field *time reversal* and *back propagation*, as we will see later in this chapter.

Example 1.3 The time-domain Green function for the one-dimensional wave equation satisfies the equation

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] g(z,t) = \delta(z)\delta(t). \tag{1.24}$$

Fourier transforming Eq. (1.24) leads to the one-dimensional Helmholtz equation