A Framework for Priority Arguments

This book presents a unifying framework for using priority arguments to prove theorems in computability. Priority arguments provide the most powerful theorem-proving technique in the field, but most of the applications of this technique are ad hoc, masking the unifying principles used in the proofs. The proposed framework presented isolates many of these unifying combinatorial principles and uses them to give shorter and easier-to-follow proofs of computability-theoretic theorems. Standard theorems of priority levels 1, 2, and 3 are chosen to demonstrate the framework's use, with all proofs following the same pattern. The last section features a new example requiring priority at all finite levels. The book will serve as a resource and reference for researchers in logic and computability, helping them to prove theorems in a shorter and more transparent manner.

Manuel Lerman is a Professor Emeritus of the Department of Mathematics at the University of Connecticut. He is the author of *Degrees of Unsolvability: Local and Global Theory*, has been the managing editor for the book series *Perspectives in Mathematical Logic*, has been an editor of *Bulletin for Symbolic Logic*, and is an editor of the ASL's *Lecture Notes in Logic* series.

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MANUEL LERMAN
University of Connecticut, Emeritus

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PREFACE

The poset (i.e., partially ordered set) of computably enumerable (i.e., recursively enumerable) degrees $\mathcal{R}$, an algebraic structure that is invariant under the notion of *information content* of sets, was introduced by Post [20] in 1944 as a way to compare the information content of computably enumerable mathematical structures and theories. It seems to have been Post's hope that $\mathcal{R}$ would provide a simple natural hierarchy based on information content, and that, perhaps, this hierarchy would consist of only two comparable degrees. Whether or not this was the case became known as Post's Problem. While this turned out not to be the case, attempts to solve Post's Problem led to an intensive ongoing study of $\mathcal{R}$.

One of the underlying themes of Computability Theory (i.e., Recursion Theory) is to determine the extent to which a given degree structure captures the notion of information content. The hope was that one could find a way to differentiate between sets with different information content within the algebraic structure. The weakest way to do so is to require that the degree structure be *rigid*, i.e., that there be no non-trivial automorphism of the structure. While the rigidity question remains unresolved, there are several results which imply that automorphisms cannot move a given degree too far.

The study of *global* properties of degrees, such as automorphisms, frequently requires an understanding of local structural properties. One way to approach a study of local structure theory is to measure the complexity of a statement as its logical complexity in a fixed language, and to try to find decision procedures for classes of sentences of a given complexity. The historical development of the subject follows an almost monotonic increase in the logical complexity of sentences about $\mathcal{R}$ whose truth is determined. This is because the primary technique used to analyze this structure, the *priority method*, consists of a uniform collection of methods of various levels of complexity, and the level of the method needed to prove that a given sentence is true is closely related to the logical complexity of the sentence. Thus the understanding of increasing levels of the priority method provided greater accessibility to results about $\mathcal{R}$ of higher logical complexity.

The mission of this book is not to present an organized analysis of the current status of knowledge about the elementary theory of $\mathcal{R}$; rather, it is to give a coherent presentation of the priority method. The framework for priority arguments developed by Steffen Lempp and the author is expanded and used to prove sentences about $\mathcal{R}$ of various complexity levels. The sentences, or theorems, chosen represent a rich cross section of results and techniques, and so demonstrate the flexibility of the framework. Furthermore, we are able to separate the technology of the priority method from the analysis necessary to prove a given result. General theorems are proved about the technology of priority arguments. This will allow us to prove theorems by first describing the action taken to satisfy single requirements, and then performing a fairly simple analysis to show that action for specified finite sets of requirements can be compatibly executed.

This monograph is an outgrowth of joint work with Steffen Lempp [10, 11, 12, 13, 14, 15], which began in 1986, and the theorems we prove frequently coincide with those presented (with standard-style proofs) in Lempp's lecture notes [9]. At that time, an attempt was made to prove the decidability of the existential theory of the poset of computably enumerable degrees in a language containing predicates representing the relations $\boldsymbol{a}^{(n)} \leq \boldsymbol{b}^{(n)}$ for all integers n. The approach taken towards the solution of that problem required priority method constructions at all levels of the arithmetical hierarchy. A general framework was developed, over the course of seven years, and used to solve the problem. It should be noted that other general frameworks were simultaneously being developed. The earliest of these was developed by Lachlan [8] in 1970 for lower levels of the arithmetical hierarchy. Particular theorems were proved in the early 1980s using priority methods at an infinite level of the hyperarithmetical hierarchy following an approach developed by Harrington. Shortly after our work began in 1986, Ash [1, 2] developed a general framework for certain types of priority constructions occurring in Computable Model Theory and covering all levels of the hyperarithmetical hierarchy. More general hierarchies of this type have since been developed by Ash and Knight (cf. [6]) but do not seem to be easily applicable to the types of constructions encountered in dealing with $\mathcal{R}$. However, some of the ideas introduced by Ash became an integral part of the framework developed jointly by Lempp and the author. Michalski [18] was able to carry out low-level priority method constructions for $\mathcal{R}$ by generalizing a framework of Ash and Knight. Another framework was under development by Groszek and Slaman, but while the ideas were general, the details were carried out only for low-level priority method constructions.

During the time that Lempp and the author were developing their original framework for the priority method, it became evident that the applicability of the framework extended beyond the theorem that was being proved. Thus in

1993, a study of the extent of applicability of the framework was begun. Initial attempts aimed at complete uniformity, but it was soon discovered that different types of requirements needed to be treated differently, so a modular approach was initiated. This approach attempted to separate the combinatorics of the priority method from the combinatorics needed to satisfy individual requirements. While this approach was successful, it required a separate description of the implementation of the framework for each construction. Furthermore, the proofs of the satisfaction of each new type of requirement made use of properties of the framework, so this approach was unattractive to the uninitiated. Many of the proofs seemed to be ad hoc in nature, and we were not comfortable with the presentation. Thus the project was temporarily suspended in 1995. When it was resumed in 1997, a modified approach that removed some of the original restrictions imposed by uniformity was discovered, thereby providing the flexibility to prove more general theorems about the framework. The approach has several steps. We begin the proof of each theorem by describing the basic module used to satisfy requirements. This module is a finite tree, with a sentence whose truth or falsity directs action at a given non-terminal node of the tree, and sentences describing the action to be taken based on the truth value of the directing sentence. The module is pasted into the highest-level tree, and a level-by-level description of the assignment of sentences to nodes of lower-level trees is presented; the new sentences are obtained from the old ones by bounding certain quantifiers. Instead of requiring the quantifier bounding algorithm to be the same for every theorem, we just list properties that must be obeyed by this algorithm. The construction takes place at the lowest-level tree, representing the computable level. We then verify that the conditions placed on assigning requirements, decomposing sentences, and specifying where the action takes place are satisfied; normally, these conditions can be verified by inspection. We appeal to theorems about the framework to complete the proof. Thus proofs are more transparent, shorter, and much easier to present.

Early in the development of the framework, goals were set for measuring its usefulness. These included:

- The approach should closely resemble the standard approach to priority method constructions using a tree of strategies.
- The approach should cover all priority method constructions used to determine properties of $\mathcal{R}$.
- The method should isolate the combinatorial lemmas which are part of the priority method from those which relate only to a restricted set of results proved by the method.
- The approach should be intuitive and helpful in finding proofs of new results.
- The approach should simplify the process of presenting proofs which can be more easily followed.

We summarize the current status of achievement of the goals, from our point of view.

- The approach uses a "system of trees of strategies" that is close to the standard "tree of strategies" approach. This should make the system of trees of strategies approach as easy to follow as other approaches to general frameworks, for those conversant with the standard approach. However, there is an additional level of abstraction that occurs in passing to a general framework and causes the approach to be less intuitive for beginners in the subject.
- We have undertaken a broad study of theorems proved by priority method constructions, and all seem to be amenable to our methods. A rich cross section of such theorems is presented in this monograph.
- We have already discussed the separation of priority from the analysis of the satisfaction of requirements. The level of separation is greater than that which was originally sought.
- Again, there is an additional level of abstraction that occurs in passing to a general framework and causes the approach to be less intuitive for beginners in the subject. We have used the approach to prove new theorems, and the intuition provided by the approach was of assistance in analyzing conflicts between strategies and finding ways to resolve them.
- We leave it to the reader to determine whether the presentation of the proofs is simpler, a subjective question. Certainly, for some proofs, shortcuts violating basic assumptions needed for the approach to succeed can be taken and used to simplify a construction. For the most part, however, the trees of strategies approach follows the same pattern as the standard approach, and the modularization makes it unnecessary to repeat the parts of the proof for which combinatorial lemmas about the framework are available. There are more combinatorial lemmas than those presented in this book, and we view the development of the framework as an ongoing process.

I would like to acknowledge my debt to Steffen Lempp for the development of the ideas in this monograph. These ideas grew out of earlier joint work with Lempp that would not have been completed without his sharp insight and persistence. His input concerning early drafts of this monograph were invaluable. Thanks are also extended to C. Ash, B. Englert, M. Groszek, L. Harrington, J. Knight, A. Kučera, R. Shore, and T. Slaman for helpful comments related to this work. Finally, I express my love and deep gratitude to my wife Maxine, and children Elliot and Sharon. Without their patience, tolerance, and understanding, this project would never have been completed.