LECTURES ON ALGEBRAIC CYCLES

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Introduction

The notes which follow are based on lectures given at Duke University in April, 1979. They are the fruit of ten years reflection on *algebraic cycles*; that is formal linear combinations $\sum n_i [V_i]$ of subvarieties V_i of a fixed smooth and projective variety X, with integer coefficients n_i.

Classically, X was an algebraic curve (Riemann surface) and the V_i were points p_i. In this context, cycles were a principal object of study for nineteenth-century complex function theory. Two (at least) of their results merit immortality: the theorems of Riemann–Roch and Abel–Jacobi.

The reader will recall that to a meromorphic function f on a smooth variety or complex manifold one can associate a *divisor* (algebraic cycle of codimension 1) (f) by taking the sum of its components of zeros minus its components of poles, all counted with suitable multiplicity. An algebraic cycle $\sum n_i [V_i]$ is said to be *effective* if all the $n_i \geq 0$. The Riemann–Roch theorem computes the dimension $\ell(D)$ of the vector space of functions f such that $(f) + D$ is effective for a given divisor D. It has been generalized quite considerably in recent years, but its central role in the study of divisors does not seem to carry over to cycles of codimension greater than 1. (For example, one can use the Riemann–Roch theorem to prove rationality of the zeta function of an algebraic curve over a finite field [9], but the corresponding theorem for varieties of dimension > 1 lies deeper.)

The other great pillar of function theory on Riemann surfaces, the Abel–Jacobi theorem, tells when a given divisor $D = \sum n_i (p_i)$ is the divisor of a function. Clearly a necessary condition is that the degree $\sum n_i$ must be zero, so we may assume this and write our cycle $D = \sum n_i ((p_i) - (p_0))$ for some base point p_0. We denote by $\Gamma(X, \Omega^1_X)$ the space of all global holomorphic differential 1-forms on X (where $\omega \in \Gamma(X, \Omega^1_X)$ can be written locally as $f(z)\,dz$ for z and $f(z)$ holomorphic). A necessary and sufficient condition for D to equal

(f) is that

$$\sum n_i \int_{p_0}^{p_i} \omega = \int_\gamma \omega$$

for some (topological) 1-cycle $\gamma \in H_1(X, \mathbf{Z})$ and all $\omega \in \Gamma(X, \Omega_X^1)$. Even more is true; writing $\Gamma(X, \Omega_X^1)^* = \mathrm{Hom}_{\mathbf{C}}(\Gamma(X, \Omega_X^1), \mathbf{C})$ for the *periods*, the map

$$A_0(X) := \frac{\text{divisors of degree } 0}{\text{divisors of functions}} \to \Gamma(X, \Omega_X^1)^*/H_1(X, \mathbf{Z}) := J(X)$$

is an isomorphism from $A_0(X)$ to the *abelian variety* $J(X)$, where $J(X)$ is the *jacobian* of X.

To extend these results, we may define for X any smooth projective variety over a field and $r \geq 0$ an integer

$$z^r(X) = \text{free abelian group generated by irreducible subvarieties of } X \text{ of codimension } r.$$

Given $A = \sum m_i A_i \in z^r(X)$ and $B = \sum n_j B_j \in z^s(X)$ such that every component of $A_i \cap B_j$ has codimension $r + s$ for all i and j, there is defined a product cycle $A \cdot B \in z^{r+s}(X)$ obtained by summing over all components of $A_i \cap B_j$ with suitable multiplicities. If $f : X \to Y$ is a proper map, where $\dim X = d$, $\dim Y = e$, one has $f_* : z^r(X) \to z^{r+e-d}(Y)$ defined on a single irreducible codimension-r subvariety $V \subset X$ by $f_*(V) = [k(V) : k(f(V))] \cdot f(V)$. Here $[k(V) : k(f(V))]$ is the degree of the extension of function fields. By convention this degree is zero if the extension is not finite.

Suppose now $\Gamma \in z^r(X \times \mathbf{P}^1)$ and no component of Γ contains either $X \times \{0\}$ or $X \times \{\infty\}$. The cycle

$$\mathrm{pr}_{1*}(\Gamma \cdot (X \times ((0) - (\infty)))) \in z^r(X), \qquad \mathrm{pr}_1 : X \times \mathbf{P}^1 \to X,$$

is then defined. By definition, $z^r_{\mathrm{rat}}(X) \subset z^r(X)$ is the subgroup of cycles of this form. Replacing $\mathbf{P}^1$ by an arbitary (variable) smooth connected curve C, and $0, \infty$ by any two points $a, b \in C$ one obtains a group $z^r_{\mathrm{alg}}(X)$ with

$$z^r_{\mathrm{rat}}(X) \subset z^r_{\mathrm{alg}}(X) \subset z^r(X).$$

We write

$$\begin{aligned} \mathrm{CH}^r(X) &= z^r(X)/z^r_{\mathrm{rat}}(X) \qquad \text{(the } \textit{Chow group}\text{)},\\ A^r(X) &= z^r_{\mathrm{alg}}(X)/z^r_{\mathrm{rat}}(X),\\ \mathrm{CH}_r(X) &= \mathrm{CH}^{d-r}(X),\\ A_r(X) &= A^{d-r}(X), \qquad d = \dim X. \end{aligned}$$

Cycles in $z^r_{\mathrm{rat}}(X)$ (resp. $z^r_{\mathrm{alg}}(X)$) are said to be *rationally* (resp. *algebraically*) *equivalent to zero* (written $x \underset{\mathrm{rat}}{\sim} 0$ or $x \underset{\mathrm{alg}}{\sim} 0$, or if no ambiguity is possible just $x \sim 0$). As an exercise, the reader might prove that when the ground field k is algebraically closed, the association $\sum n_i(p_i) \mapsto \sum n_i$ defines an isomorphism $\mathrm{CH}_0(X)/A_0(X) \cong \mathbf{Z}$. A bit more difficult is to show

$$\mathrm{CH}^1(X) \cong \frac{\text{divisors}}{\text{divisors of functions}} = \text{divisor class group of } X \cong \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. Still harder is to prove that when X has a rational point, $A^1(X)$ is isomorphic to the group of k-points of the *Picard variety* of X.

The purpose of these notes is to study algebraic cycles, *particularly those of codimension* > 1 as it is here that really new and unexpected phenomena occur. In keeping with the author's philosophy that good mathematics "opens out" and involves several branches of the subject, we will consider geometric, algebraic and arithmetic problems.

The first three lectures are geometric in content, studying various aspects of the Abel–Jacobi construction in codimensions > 1. We work with varieties X over the complex numbers, and define (following Griffiths, Lieberman, and Weil) compact complex tori (*intermediate jacobians*) $J^r(X)$ and *Abel–Jacobi maps*

$$\Theta\colon A^r(X) \to J^r(X).$$

When $r = d = \dim X$, $J^r(X) = J_0(X) = \Gamma(X, \Omega^1_X)^*/H_1(X, \mathbf{Z})$ is called the Albanese variety and the Abel–Jacobi map is defined precisely as with curves. The map $\Theta\colon A_0(X) \to J_0(X)$ is surjective, and in the first lecture we study its kernel $T(X)$ when $\dim X = 2$. We sketch an argument of Mumford that $T(X) \neq 0$ (in fact $T(X)$ is enormous) when $P_g(X) \neq 0$, i.e. when X has a non-zero global holomorphic 2-form. We give several examples motivating the conjecture that $\Gamma(X, \Omega^2_X)$ actually controls the structure of $T(X)$ and that in particular $T(X) = 0 \iff \Gamma(X, \Omega^2_X) = 0$. This conjecture can be formulated in various ways. One vague but exciting possibility is that groups like $T(X)$ provide a geometric interpretation of the category of *polarized Hodge structures of weight two*, in much the same way that abelian varieties do for weight 1.

The second and third lectures consider curves on threefolds. In Lecture 2 we focus on quartic threefolds, i.e. smooth hypersurfaces X of degree 4 in $\mathbf{P}^4$, and verify in that case Θ is an isomorphism. Lecture 3 considers *relative* algebraic 1-cycles on $X = C \times \mathbf{P}^1 \times \mathbf{P}^1$, where C is a curve. We define a *relative* intermediate jacobian which turns out to be a non-compact torus isomorphic to $H^1(C, \mathbf{C}^*)$. The machinery of cycle classes constructed in this lecture is related in Lectures 8 and 9 to special values of Hasse–Weil zeta functions. Of particular importance are the classes in $H^1(C, \mathbf{R})$ defined by factoring out by the maximal compact. We show such classes can be defined for any (not necessarily relative) cycle on $C \times \mathbf{P}^1 \times \mathbf{P}^1$.

Lectures 4 through 6 develop the algebraic side of the theory: the cohomology groups of the K-sheaves $H^p(X, \mathcal{K}_q)$, the Gersten–Quillen resolution, and analogues for singular and étale cohomology theories. Of particular interest are the cohomological formulae

$$\mathrm{CH}^p(X) \cong H^p(X, \mathcal{K}_p)$$

and (for X defined over $\mathbf{C}$)

$$\mathrm{CH}^p(X)/A^p(X) \cong H^p(X, \mathcal{H}^p),$$

where $\mathcal{H}^p$ denotes the sheaf for the Zariski topology associated to the presheaf $U \subset X \rightarrow H^p(U, \mathbf{Z})$ (singular cohomology). These techniques are used to prove a theorem of Roitman that $A_0(X)_{\mathrm{tors}}$, the torsion subgroup of the zero-cycles, maps isomorphically to the torsion subgroup of the Albanese variety. The heart of the proof is a result about the multiplicative structure of the Galois cohomology *ring* of a function field F of transcendence degree d over an algebraically closed field. We prove for ℓ prime to char F that the cup product map

$$\underbrace{H^1(F, \mathbf{Z}/\ell\mathbf{Z}) \otimes \cdots \otimes H^1(F, \mathbf{Z}/\ell\mathbf{Z})}_{d \text{ times}} \rightarrow H^d(F, \mathbf{Z}/\ell\mathbf{Z})$$

is surjective. It would be of great interest both algebraically and geometrically to know if the whole cohomology ring $H^*(F, \mathbf{Z}/\ell\mathbf{Z})$ were generated by H^1.

The last three lectures are devoted to arithmetic questions. In Lecture 7 we consider $A_0(X)$, where X is a surface over a local or global field k. We assume the base extension of X to the algebraic closure of k is a rational surface. Using a technique of Manin involving the *Brauer group* we show by example that in general $A_0(X) \neq 0$. We prove that $A_0(X)$ is finite when X has a pencil of genus-zero curves (*conic bundle surface*). The key idea in the proof is a sort of generalization of the *Eichler norm theorem*, describing the image of the reduced norm map $\mathrm{Nrd}\colon A^* \rightarrow k(t)^*$ when A is a quaternion algebra defined over a rational field in one variable over k.

Finally, Lectures 8 and 9 take up, from a number-theoretic point of view, the work of Lecture 3 on relative intermediate jacobians for curves on $C \times \mathbf{P}^1 \times \mathbf{P}^1$. We consider the case $C = E =$ elliptic curve and compute explicitly the class in $H^1(E, \mathbf{R})$ associated to the curve

$$\gamma_{f,g} = \{(x, f(x), g(x)) \mid x \in E\}$$

for f, g rational functions on E. When the zeros and poles of f are points of finite order, we show how to associate to f and g an element in $\Gamma(E, \mathcal{K}_2)$ and how to associate to an element in $\Gamma(E, \mathcal{K}_2)$ a relative algebraic 1-cycle on $E \times \mathbf{P}^1 \times \mathbf{P}^1$. When E has complex multiplication by the ring of integers in an imaginary quadratic field of class number one, we construct an element $U \in \Gamma(E_{\mathbf{Q}}, \mathcal{K}_2)$ such that the image under the Abel–Jacobi map into $H^1(E, \mathbf{R}) \cong \mathbf{C}$ of the associated relative algebraic cycle multiplied by a certain simple constant (involving the conductor of the curve and Gauss sum) is the value of the Hasse–Weil zeta function of E at $s = 2$. Conjecturally for E defined over a number field k, the rank of $\Gamma(E_k, \mathcal{K}_2)$ equals the order of the zero of the Hasse–Weil zeta function of E at $s = 0$.

I want to thank Duke University for financial support, and my auditors at Duke for their enthusiasm and tenacity in attending eight lectures in ten days. I also want to acknowledge that much of the function theory of the dilogarithm which underlines the calculations in the last two lectures was worked out in collaboration with David Wigner, and that I never would have gotten the damned constants in (9.12) correct without help from Dick Gross. (I may, indeed, not have gotten them correct even with help.)

Finally, a pedagogical note. When an idea is already well documented in the literature and extensive detail would carry us away from the focus of these notes on algebraic cycles, I have been very sketchy. For example, Quillen's work on the foundations of K-theory are magnificently presented in his own paper [10]. I only hope the brief outline given here will motivate the reader to turn to that source. The reader may also find the rapid treatment of the Mumford argument showing $P_g \neq 0 \Rightarrow T(X) \neq 0$ in Lecture 1 to be unsatisfactory. To remedy this, another demonstration rather different in spirit from Mumford's is included as an appendix. Historically, the negative force of this result led us all to conclude that, except in certain obvious cases such as ruled surfaces, the structure of zero-cycles on a surface was total chaos. I wanted to devote time to various examples showing that this is not the case. In retrospect, I have certainly "left undone those things which I ought to have done" (e.g. Griffiths' proof that homological equivalence $\neq$ algebraic equivalence, and Tate's proof of the Tate conjecture for abelian varieties). I hope the reader will spare me.

References for Lecture 0

A list of references appears at the end of each lecture. What follows are references for foundational questions about cycles, etc.

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1

Zero-cycles on surfaces

In this first section I want to consider the question of zero-cycles on an algebraic surface from a purely geometric point of view. I will consider a number of explicit examples, and give a heuristic description of a result of Mumford [4] that $P_g \neq 0$ implies $A_0(X)$ is "very large". In particular $A_0(X)$ is not an abelian variety in this case. Finally I will discuss some conjectures motivated by these ideas.

For purposes of this lecture, algebraic surface X will mean a smooth projective variety of dimension 2 over the complex numbers (or, if you prefer, a compact complex manifold of dimension 2 admitting a projective embedding). The space of global holomorphic i-forms ($i = 1, 2$) will be written $\Gamma(X, \Omega^i_X)$ (or $H^{i,0}$) and we will write

$$q = \dim \Gamma(X, \Omega^1_X), \quad P_g = \dim \Gamma(X, \Omega^2_X).$$

If γ is a topological 1-chain on X, the integral over γ, $\int_\gamma$, is a well-defined element in the dual $\mathbf{C}$-vector space $\Gamma(X, \Omega^1_X)^*$. If C is a 2-chain,

$$\int_{\partial C} \omega = \int_C d\omega \qquad \text{(Stokes')}$$

so $\int_{\partial C} = 0$ in $\Gamma(X, \Omega^1_X)^*$, since global holomorphic forms are closed. Hence we may map $H_1(X, \mathbf{Z}) \to \Gamma(X, \Omega^1_X)^*$. It follows from the Hodge-theoretic decomposition $H^1(X, \mathbf{C}) = H^{0,1} \oplus H^{1,0}$ that the quotient $\Gamma(X, \Omega^1_X)^*/H_1(X, \mathbf{Z})$ is a compact complex torus, called the Albanese of X and written $\mathrm{Alb}(X)$. This torus admits a polarization satisfying the Riemann bilinear relations and hence is an abelian variety.

Fixing a base point $p_0 \in X$, we define $\phi\colon X \to \mathrm{Alb}(X)$ by $\phi(p) = \int_{p_0}^{p}$. Note that this is well defined: two paths from p_0 to p differ by an element in $H_1(X, \mathbf{Z})$. Notice, finally, that the above discussion is equally valid for smooth projective varieties X of arbitrary dimension.

Proposition (1.1)

(i) Let $\phi_n\colon \underbrace{X \times \cdots \times X}_{n \text{ times}} \to \mathrm{Alb}(X)$ *be defined by* $\phi_n(x_1, \ldots, x_n) = \phi(x_1) + \cdots + \phi(x_n)$. *Then for* $n \gg 0$, ϕ_n *is surjective.*

(ii) Let $\psi\colon X \to A$ *be a map from* X *to a complex torus, and assume* $\psi(p_0) = 0$. *Then there exists a unique homomorphism* $\theta\colon \mathrm{Alb}(X) \to A$ *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mathrm{Alb}(X) \arrow[dl, dashed, "\theta"] \\ A & \end{tikzcd}$$

commutes.

(iii) The map $z_0(X) \xrightarrow{\phi} \mathrm{Alb}(X)$, *where* $(x) \mapsto \phi(x)$, *factors through the map* $\phi\colon \mathrm{CH}_0(X) \to \mathrm{Alb}(X)$. *The induced map* $\phi\colon A_0(X) \to \mathrm{Alb}(X)$ *is surjective, and independent of the choice of base point* p_0.

Proof These results are more or less well known. The reader who is unfamiliar with them might try as an exercise to find proofs. (Hint: In (i) consider the question infinitesimally and use the fundamental theorem of calculus to calculate the derivative of $\int_{p_0}^{p} \omega$. For (iii), reduce the question to showing that a map $\mathbf{P}^1 \to$ complex torus is necessarily constant.) $\square$

Example (1.2) (Bloch et al. [2]) Let E and F be elliptic curves (Riemann surfaces of genus 1). We propose to calculate the Chow group of the quotient surface $X = (F \times E)/\{1, \sigma\}$, where σ is a fixed-point-free involution on $F \times E$ obtained by fixing a point $\eta \in E$ of order 2, $\eta \neq 0$, and taking $\sigma(f, e) = (-f, e + \eta)$. Let $E' = E/\{1, \eta\}$. There is a natural map $\rho : X \to E'$ with all fibres of $\rho \cong F$.

Notice

$$\begin{aligned} \Gamma(X, \Omega^1_X) &\cong \Gamma(F \times E, \Omega^1_{F \times E})^{\{1,\sigma\}} \\ &\cong \left[\Gamma(E, \Omega^1_E) \oplus \Gamma(F, \Omega^1_F)\right]^{\{1,\sigma\}} \\ &\cong \Gamma(E, \Omega^1_E) \cong \mathbf{C} \end{aligned}$$

since the automorphism $f \to -f$ acts by -1 on $\Gamma(F, \Omega^1_F)$. We conclude $\mathrm{Alb}(X)$ has dimension 1. Since $\mathrm{Alb}(X) \to E'$ and the fibres of ρ are connected, it follows that $\mathrm{Alb}(X) \cong E'$.

Lemma (1.3) *Let* Y *be a smooth quasi-projective variety,* $n > 0$ *an integer. Then* $A_n(X)$ *is a divisible group.*

Proof By definition $A_n(X) \subset \mathrm{CH}_n(X)$ is generated by images under correspondences from jacobians of curves. Since jacobians of curves are divisible groups, the lemma follows. □

Lemma (1.4) *Let Y be a smooth projective variety, and let*

$$T(Y) = \mathrm{Ker}\,(A_0(Y) \to \mathrm{Alb}(Y)).$$

Then $T(Y)$ is divisible.

Proof For any abelian group A, let ${}_NA \subset A$ be the kernel of multiplication by N. From the divisibility of $A_0(Y)$ one reduces to showing ${}_NA_0(Y) \twoheadrightarrow {}_N\mathrm{Alb}(Y)$ for any N. If Y is a curve, the Chow group and Albanese both coincide with the jacobian so ${}_NA_0(Y) \cong {}_N\mathrm{Alb}(Y)$. It will suffice therefore to assume $\dim Y > 1$ and show ${}_N\mathrm{Alb}(W) \twoheadrightarrow {}_N\mathrm{Alb}(Y)$ for $W \subset Y$ a smooth hyperplane section. As a real torus, $\mathrm{Alb}(Y)$ can be identified with $H_1(Y, \mathbf{R}/\mathbf{Z})$ and ${}_N\mathrm{Alb}(Y) = H_1(Y, \mathbf{Z}/N\mathbf{Z})$, so the question becomes the surjectivity of $H_1(W, \mathbf{Z}/N\mathbf{Z}) \to H_1(Y, \mathbf{Z}/N\mathbf{Z})$ or the vanishing of $H_1(Y, W; \mathbf{Z}/N\mathbf{Z})$. This group is identified by duality with $H^{2\dim Y-1}(Y - W, \mathbf{Z}/N\mathbf{Z}) = (0)$ because the Stein variety $Y - W$ has cohomological dimension $= \dim Y$. □

We now return to our surface $X = F \times E/\{1, \sigma\}$.

Claim $A_0(X) \cong E' \cong \mathrm{Alb}(X)$. *That is,* $T(X) = 0$.

Proof We work with the diagram

$$\begin{tikzcd}
F \times E \arrow[r] \arrow[d, "\pi"'] & E \arrow[d] \\
X \arrow[r] & E'.
\end{tikzcd}$$

If $z \in T(X)$ we may write

$$\pi^* z = \sum r_i\,[(q_i, p_i) + (-q_i, p_i + \eta)],$$

where $\sum r_i = 0$ and $\sum 2\,r_i p_i = 0$ in E. By Abel's theorem

$$\begin{aligned} 2\,(q, p) &\sim 2\,(q, p + \eta) \qquad \text{on} \;\; F \times E, \\ (q, p) + (-q, p) &\sim 2\,(0, p), \end{aligned}$$

so

$$\begin{aligned} 2\,\pi^* z \sim \sum 2\,r_i\,[(q_i, p_i) + (-q_i, p_i)] &\sim \sum 4\,r_i\,(0, p_i) \\ &= [(0) \times E] \cdot \Big[F \times \sum 4\,r_i\,(p_i)\Big] \\ &\sim 0\,. \end{aligned}$$