Introduction: In Which Mathematics Sets Out to Conquer New Territories

IT'S BEEN said again and again: the century that just ended was the true golden age of mathematics. Mathematics evolved more in the twentieth century than in all previous centuries put together. Yet the century just begun may well prove exceptional for mathematics, too: the signs seem to indicate that, in the coming decades, mathematics will undergo as many metamorphoses as in the twentieth century – if not more. The revolution has already begun. From the early seventies onward, the mathematical method has been transforming at its core: the notion of proof. The driving force of this transformation is the return of an old, yet somewhat underrated mathematical concept: that of computing.

The idea that computing might be the key to a revolution may seem paradoxical. Algorithms that allow us, among other things, to perform sums and products are already recognized as a basic part of mathematical knowledge; as for the actual calculations, they are seen as rather boring tasks of limited creative interest. Mathematicians themselves tend to be prejudiced against computing – René Thom said: "A great deal of my assertions are the product of sheer speculation; you may well call them reveries. I accept this qualification.... At a time when so many scientists around the world are computing, should we not encourage those of them who can to dream?" Making computing food for dreams does seem a bit of a challenge.

Unfortunately, this prejudice against computing is ingrained in the very definition of mathematical proof. Indeed, since Euclid, a proof has been defined as reasoning built on axioms and inference rules. But are mathematical problems always solved using a reasoning process? Hasn't the practice of mathematics shown, on the contrary, that solving a problem requires the subtle arrangement of reasoning stages and computing stages? By confining itself to reasoning, the axiomatic method may offer only a limited vision of mathematics. Indeed, the axiomatic method has reached a crisis, with recent mathematical advances, not all related to one another, gradually challenging the primacy of reasoning over computing and suggesting a more balanced approach in which these two activities play complementary roles.

This revolution, which invites us to rethink the relationship between reasoning and computing, also induces us to rethink the dialogue between mathematics and natural sciences such as physics and biology. It thus sheds new light on the age-old question of mathematics's puzzling effectiveness in those fields, as well as on the more recent debate about the logical form of natural theories. It prompts us to reconsider certain philosophical concepts such as analytic and synthetic judgement. It also makes us reflect upon the links between mathematics and computer science and upon the singularity of mathematics, which appears to be the only science where no tools are necessary.

Finally, and most interestingly, this revolution holds the promise of new ways of solving mathematical problems. These new methods will shake off the shackles imposed by past technologies that have placed arbitrary limits on the lengths of proofs. Mathematics may well be setting off to conquer new, as yet inaccessible territories.

Of course, the crisis of the axiomatic method did not come out of the blue. It had been heralded, from the first half of the twentieth century, by many signs, the most striking being two new theories that, without altogether questioning the axiomatic method, helped to reinstate computing in the mathematical edifice, namely the theory of computability and the theory of constructivity. We will therefore trace the history of these two ideas before delving into the crisis. However, let us first head for remote antiquity, where we will seek the roots of the very notion of computing and explore the "invention" of mathematics by the ancient Greeks.

PART ONE

Ancient Origins

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More information

CHAPTER ONE

The Prehistory of Mathematics and the Greek Resolution

THE ORIGIN OF MATHEMATICS is usually placed in Greece, in the fifth century B.C., when its two main branches were founded: arithmetic by Pythagoras and geometry by Thales and Anaximander. These developments were, of course, major breakthroughs in the history of this science. However, it does not go far enough back to say that mathematics has its source in antiquity. Its roots go deeper into the past, to an important period that laid the groundwork for the Ancient Greeks and that we might call the "prehistory" of mathematics. People did not wait until the fifth century to tackle mathematical problems – especially the concrete problems they faced every day.

ACCOUNTANTS AND LAND SURVEYORS

A tablet found in Mesopotamia and dating back to 2500 B.C. carries one of the oldest traces of mathematical activity. It records the solution to a mathematical problem that can be stated as follows: if a barn holds 1,152,000 measures of grain, and you have a barn's worth of grain, how many people can you give seven measures of grain to? Unsurprisingly, the result reached is 164,571 – a number obtained by dividing 1,152,000 by seven – which proves that Mesopotamian accountants knew how to do division long before arithmetic was born. It is even likely (although it is hard to know anything for certain in that field) that writing was invented in order to keep account books and that, therefore, numbers were invented before letters. Though it may be hard to stomach, we probably owe our whole written culture to a very unglamorous activity: accounting.

Mesopotamian and Egyptian accountants not only knew how to multiply and divide, but had also mastered many other mathematical operations – they were able to solve quadratic equations, for instance. As for land surveyors, they knew how to measure the areas of rectangles, triangles, and circles.

THE IRRUPTION OF THE INFINITE

The techniques worked out by accountants and surveyors constitute the prehistory of arithmetic and geometry; as for the history of mathematics, it is considered to have begun in Ancient Greece, in the fifth century B.C. Why was this specific period chosen? What happened then that was so important? In order to answer this question, let us look at a problem solved by one of Pythagoras's disciples (whose name has been forgotten along the way). He was asked to find an isosceles right triangle whose three sides each measured a whole number of a unit – say, meters. Because the triangle is isosceles, its two short sides are the same length, which we will call x. The length of the hypotenuse (i.e. the triangle's longest side) we will call y. Because the triangle is right, y^2, according to Pythagoras's theorem, equals $x^2 + x^2$. To look for our desired triangle, then, let's try all possible combinations where y and x are less than 5:

x	y	$2 \times x^2$	y^2
1	1	2	1
1	2	2	4
1	3	2	9
1	4	2	16
2	1	8	1
2	2	8	4
2	3	8	9
2	4	8	16
3	1	18	1
3	2	18	4
3	3	18	9
3	4	18	16
4	1	32	1
4	2	32	4
4	3	32	9

In all these cases, the number $2 \times x^2$ is different from y^2. We could carry on searching, moving on to larger numbers. In all likelihood, Pythagoras's followers kept looking for the key to this problem for a long time, in vain, until they eventually became convinced that no such triangle existed. How did they manage to reach this conclusion, namely that the problem could not be solved? Not by trying out each and every pair of numbers one after the other, for there are infinitely many such pairs. Even if you tried out all possible pairs up to one thousand, or even up to one million, and found none that worked, you still could not state with any certainty that the problem has no solution – a solution might lie beyond one million.

Let's try to reconstruct the thought process that may have led the Pythagoreans to this conclusion.

First, when looking for a solution, we can restrict our attention to pairs in which at least one of the numbers x and y is odd. To see why, observe that if the pair $x = 202$ and $y = 214$, for example, were a solution, then, by dividing each number by two, we would find another solution, $x = 101$ and $y = 107$, where at least one of the numbers is odd. More generally, if you were to pick any solution and divide it by two, repeatedly if necessary, you would eventually come to another solution in which at least one of the numbers is odd. So, if the problem has any solution, there is necessarily a solution in which either x or y is an odd number.

Now, let's divide all pairs of numbers into four sets:

- pairs in which both numbers are odd;
- pairs in which the first number is even and the second number is odd;
- pairs in which the first number is odd and the second number is even;
- pairs in which both numbers are even.

We can now give four separate arguments to show that none of these sets holds a solution in which at least one of the numbers x and y is odd. As a result, the problem cannot be solved.

Begin with the first set: it cannot contain a solution in which one of the numbers x and y is odd, because if y is an odd number, then so is y^2; as a consequence, y^2 cannot equal $2 \times x^2$, which is necessarily an even number. This argument also rules out the second set, in which x is even and y odd. Obviously, the fourth set must also be ruled out, because by definition it cannot contain a pair where at least one number is odd. Which leaves us with the third set. In this case, x is odd and y is even, so that the number obtained by halving $2 \times x^2$ is odd, whereas half of y^2 is even – these two numbers cannot be equal.

The conclusion of this reasoning, namely that a square cannot equal twice another square, was reached by the Pythagoreans more than twenty-five centuries ago and still plays an important part in contemporary mathematics. It shows that, when you draw a right isosceles triangle whose short side is one meter long, the length of the hypotenuse measured in meters is a number (slightly greater than 1.414) that cannot be obtained by dividing x and y, two natural numbers, by each other. Geometry thus conjures up numbers that cannot be derived from integers using the four operations – addition, subtraction, multiplication, and division.

Many centuries later, this precedent inspired mathematicians to construct new numbers, called "real numbers." The Pythagoreans, however, did not go quite so far: they were not ready to give up what they regarded as the essential value of natural numbers. Their discovery felt to them more like a disaster than an opportunity.

Yet the Pythagorean problem was revolutionary not only because of its effects, but also because of how it is framed and how it was solved. To begin with, the Pythagorean problem is much more abstract than the question found on the Mesopotamian tablet, where 1,152,000 measures of grain were divided by 7 measures. Whereas the Mesopotamian question deals with measures of grain, the Pythagorean problem deals with numbers and nothing more. Similarly, the geometric form of the Pythagorean problem does not concern triangular fields but abstract triangles. Moving from a number of measures of grain to a number, from a triangular field to a triangle, may seem a trifle, but abstraction is actually a step of considerable importance. A field cannot measure more than a few kilometers. If the problem involved an actual triangular field, it would suffice, in order to solve it, to try every solution in which x and y are less than 10,000. But, unlike a triangular field, an abstract triangle can easily measure a million units, or a billion, or any magnitude.

Clearly, a rift had opened between mathematical objects, which are abstract, and concrete, natural objects – and this exists even when the mathematical objects have been abstracted from the concrete ones. It is this rift that was the big breakthrough of the fifth century B.C.

The growing distance between mathematical objects and natural ones led some people to think that mathematics was not fit to describe natural objects. This idea dominated until the seventeenth century – Galileo's day – when it was refuted by advances in mathematical physics. Yet it persists today in those views that deny mathematics any relevance in the fields of social sciences – as when Marina Yaguello argues that the role of mathematics in linguistics is to "cover up its 'social' (hence fundamentally inexact) science with complex formulae."

This change in the nature of the objects under study – which, since the fifth century B.C., have been geometric figures and numbers not necessarily related to concrete objects – triggered a revolution in the method used to solve mathematical problems. Once again, let's compare the methods used by the Mesopotamians and those used by the Pythagoreans. The tablet shows that Mesopotamians solved problems by performing computations – to answer the question about grain, they did a simple division. When it comes to the Pythagoreans' problem, however, reasoning is necessary.

In order to do a division, all you have to do is apply an algorithm taught in primary school, of which the Mesopotamians knew equivalents. By contrast, when developing their thought process, the Pythagoreans could not lean on any algorithm – no algorithm recommends that you group the pairs into four sets. To come up with this idea, the Pythagoreans had to use their imaginations. Maybe one of Pythagoras's followers understood that the number y could not be odd and then, a few weeks or a few months later, another disciple helped make headway by discovering that x could not be an odd number either. Perhaps it was months or even years before another Pythagorean made the next big advance. When a Mesopotamian tackled a division, he knew he was going to achieve a result. He could even gauge beforehand how long the operation would take him. A Pythagorean tackling an arithmetic problem had no means of knowing how long it would be before he found the line of reasoning that would enable him to solve the problem – or even if he ever would.

Students often complain that mathematics is a tough subject, and they're right: it is a subject that requires imagination; there is no systematic method for solving problems. Mathematics is even more difficult for professional mathematicians – some problems have remained unsolved for decades, sometimes centuries. When trying to solve a math problem, there is nothing unusual about drawing a blank. Professional mathematicians often stay stumped too, sometimes for years, before they have a breakthrough. By contrast, no one dries up over a division problem – one simply commits the division algorithm to memory and applies it.

How did the change in the nature of mathematical objects bring about this methodological change? In other words, how did abstraction lead mathematicians to drop calculation in favor of the reasoning that so characterizes Ancient Greek mathematics? Why couldn't the Pythagorean problem be solved by simple calculation? Think back, once more, to the Mesopotamian question. It deals with a specific object (a grain-filled barn) of known size. In the Pythagorean problem, the size of the triangle is not known – indeed, that's the whole problem. So the Pythagorean problem does not involve a specific triangle but, potentially, all possible triangles. In fact, because there is no limit to the size a triangle might reach, the problem concerns an infinity of triangles simultaneously. The change in the nature of the objects being studied is thus accompanied by the irruption of the infinite into mathematics. It was this irruption that made a methodological change necessary and required reasoning to be substituted for computing. For, if the problem concerned a finite number of triangles – for example, all triangles whose sides measure less than 10,000 metres – we could still resort to calculation. Trying out every possible pair of whole numbers up to 10,000 would doubtless be tedious without the aid of a machine, but it is nonetheless systematic and would settle the finite problem. As we've observed, though, it would be futile against the infinite.

This is why the transition from computing to reasoning, in the fifth century B.C. in Greece, is regarded as the true advent of mathematics.