Introduction

But at some point it is necessary to go back again to the foundations and, this time, observe complete rigour.

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We define a thin shell as a body bounded by two closely spaced curved surfaces. Assume that every point of the shell is associated with the curvilinear coordinates $\alpha_1, \alpha_2$ and the unit normal vector $\hat{m}$, such that the distance along $\hat{m}$ is given by $z (-0.5h(\alpha_1, \alpha_2) \leq z \leq 0.5h(\alpha_1, \alpha_2))$ (Fig. 1). Then this body is called the shell of thickness $h$. Let the faces of the shell be smooth with no singularities. The shell is classified as thin or thick on the basis of the ratio $h/R_i$, where $R_i$ are the radii of curvature of the middle surface $S$ of the shell, i.e. the surface at $z=0$. Thus, the shell is considered to be thin if $\max(h/R_i) \leq 1/20$ and thick otherwise. However, it should be noted that the above estimate is very rough and in many practical applications other geometrical and mechanical characteristics should also be considered.

Most organs of the human body, including the eyeball, oesophagus, stomach, gallbladder, uterus, ureter and bladder, can be viewed as thin shells. Their high endurance and enormous functionality depend on biomechanical properties of the tissues they are made of and specific arrangements of constituents (proteins, fibrils, cells) within them. Biological tissues are regarded as anisotropic, heterogeneous, incompressible composites. They are inherently nonlinear in their mechanical response and undergo finite deformations. Additionally, most biological tissues, with the exception of bones, are soft. Thus, they (i) are thin, (ii) possess low stiffness in response to elongation, (iii) do not resist compression and bending, (iv) undergo large deformations, (v) generate lateral (shear) stresses that are small compared with tangent stresses, and (vi) may wrinkle during operation without loss of stability of the organ. The above properties define the high degree of variability of shapes that the organ can take on in the process of loading.
The distinctive anatomical appearance of organs is also correlated with their structural advantages. They contain the optimal space within and outside, exhibit high degrees of reserved strength and structural integrity combined with efficient biomechanical functionality, have optimal strength-to-weight ratios and are ideal to resist (support) the internal pressure and external loads. For example, the human stomach is the organ of the gastrointestinal tract located in the left upper quadrant of the abdomen. Its prime role is to accommodate and digest food. Even with small thickness of the gastric wall, which in normal subjects varies from 3 to 5 mm, and the characteristic radius of curvature of the middle surface within the range $10 \, \text{cm} \leq R_i \leq 15 \, \text{cm}$ it is capable of holding $2–5 \, \text{l}$ of mixed gastric content without increasing the intraluminal pressure.

The pregnant uterus is the organ of pear-like shape that occupies the lower and middle abdomen. Its prime functions are to accommodate and nurture the fetus (fetuses) during gestation, and to expel the baby during labour and delivery. The thickness of the uterine wall in different regions varies in the range $0.5–1.5 \, \text{cm}$ and the radii of curvature vary in the range $20–40 \, \text{cm}$. Therefore, the pregnant uterus can also be approximated as a thin soft shell.

With the latest advances in mathematical modelling of biological systems, it has become possible to develop complex models of the abdominal organs and to gain insight into the hidden physiological mechanisms of their function (Miftahof et al., 2009; Pullan et al., 2004; Cheng et al., 2007; Corrias and Buist, 2007; Pal et al., 2004; Pal et al., 2007). A first biomechanical model of the organ as a soft biological shell was developed by Miftakhov (1983c). Under general assumptions of curvilinear orthotropy and physical and geometrical nonlinearity, a mathematical formulation and a numerical investigation of the dynamics of stress–strain distribution in the organ under simple and complex loadings were performed. The dynamics of the development of uniaxial stress–strained states in the cardia and pylorus as a function of intraluminal pressure was demonstrated computationally and confirmed experimentally. The results provided a valuable
insight into the mechanism of blunt abdominal trauma with rupture of the anterior wall of the stomach and gave a biomechanical explanation for the Mallory–Weiss syndrome. It was thought previously that atrophic changes in the gastric mucosa and submucous layer were responsible for longitudinal tears in the cardia-fundal region and life-threatening intragastric bleeding. The soft-shell-model studies demonstrated that the anatomical structure and configuration of the stomach per se make these regions more susceptible than the others to linear ruptures.

The biomechanics of the small intestine has been extensively studied experimentally and numerically. Miftahof was the first to construct a biophysically plausible model of the organ as a soft cylindrical biological shell. With the model it was possible to reproduce a variety of electromechanical wave phenomena, including the gradual reflex, pendular movements, segmentation and peristalsis. The model also contained intrinsic neuroregulatory mechanisms – the enteric nervous plexuses and multiple neurotransmitters. Thus, the model allowed one to study the effects of different classes of pharmacological compounds on the motility of the small intestine in normal and pathological conditions.

Mathematical models of visceral organs addressing various aspects of physiological functions have been proposed recently. However, all models, without exception, are based on a reductionist ‘mechanistic’ approach and thus have limited biological plausibility and implications for our understanding of the pathophysiology of various diseases. This is often due to indiscrete and erroneous applications of ideas and methods borrowed from the mechanics of solids to describe their mechanical behaviour. It should be emphasized that biomechanics is not just the transformation of general laws and principles of mechanics to the study of biological phenomena, but rather the adequate development and extension of these laws and principles to the modelling and analysis of living things. Therefore, accurate integrative models that incorporate various data and serve as the basis for multilevel analysis of interrelated biological processes are required. Such models will have enormous impact on unravelling hidden intricate mechanisms of diseases and assist in the design of their treatment.

Unfortunately, currently it is still common practice in the community of modellers to employ the system of Navier–Stokes equations when modelling the hollow abdominal viscera as a ‘shell’ structure. Using commercially available software and highly flexible graphical tools, they manage to fit results of numerical simulations to experimental data. The approach is utterly incorrect and the results cause confusion, rather than providing solutions to urgent clinical problems. Thus, it is erroneous to claim that antral contraction wave activity plays the dominant role in intragastric fluid motions, on the basis of results of computer simulations of a flow caused by prescribed indentations of the surface boundaries (Pal et al., 2004; Pal et al., 2007). The latter system is supposed to
represent a two-dimensional model of the stomach. It is not surprising that the
numerical method, which predetermines the desired patterns of flow, produces
results that resemble those observed experimentally, in studies using magnetic
resonance imaging. An adequate mathematical model of the above phenomenon
should have comprised the combined system of the equations of motion of the
bioshell – the stomach – and Navier–Stokes equations for the gastric content.
Also, in view of the fundamental mechanical property of the tissue, its softness, it is
unwise to argue for the dependence of stress–strain states on the radii of curvature of
visceral organs (Liao et al., 2004). It is the responsibility of an applied mathema-
tician, a computer scientist and a mechanical engineer to suggest an adequate
descriptor and to give a rigorous mathematical formulation of the model.

Although the models of the stomach and of the small and large intestine as
biological shells described in this book have limited biomedical value, they are
mathematically sound and are based on the accurate extension and application of
general laws and hypotheses of the mechanics of thin soft shells. They incorporate
electrophysiological and morphological data concerning the structure and function
of human organs and reproduce quantitatively and qualitatively the dynamics of
electromechanical wave activity and the stress–strain distribution in them. They can
serve as a starting point for further expansions and biological improvements. We
hope that with the publication of this book the approach to modelling of soft
abdominal organs will be reconsidered.

Exercises

1. As noted once by Charles Darwin (1809–1882): ‘I deeply regretted that I did not proceed
far enough at least to understand something of the great leading principles of mathe-
matics; for men thus endowed seem to have an extra sense’. Why is mathematical
modelling in the life sciences so hard?
2. ‘Reduction’ versus ‘integration’ is the continual dilemma in mathematical modelling.
Recall Albert Einstein (1879–1955): ‘Models should be as simple as possible, but not
more so’. How far should we go to achieve the balance?
3. Biology/medicine is an empirical science; nothing is ever proven. Explanations are given
in terms of the concepts and prevailing perspectives of the time and available experi-
mental facts regarding a particular phenomenon. What is the role of the mathematical
sciences in medicine and biology?
4. The biological phenomena that an investigator seeks to understand and predict are very
rich and diverse. They are not derived from a few simple principles. Should a mathe-
matical modeller look for specific biological laws or learn to apply general laws of nature
to study biological processes?
5. Biological systems are large and complex. Their dynamics can be hard to understand by
intuitive approaches alone. Systems biology is a new paradigm that offers a holistic
approach to the investigation of interrelations and interactions at various structural levels of the system. What elements are essential in the study of the biomechanics of the digestive system, i.e. the stomach and the small and large intestine?

6. When does a mathematical model become satisfactory and useful? Formulate some criteria of a satisfactory mathematical model.

7. What is required in order for one to be a mathematical biologist?

8. Researchers rely on conventional solvers – commercially available packages such as MATLAB, BEM, MAPLE, etc. – in situations involving nonlinear systems of differential equations. However, one should be aware of imminent pitfalls that might not always be easy to recognize when dealing with mathematical formulations that are different from the classical ones. What problems should a modeller be aware of when using conventional software in solving new mathematical problems?
The geometry of the surface

1.1 Intrinsic geometry

Consider a smooth surface $S$ in three-dimensional Euclidean space. It is referred to a right-handed global orthogonal Cartesian system $x_1, x_2, x_3$. Let $S$ also be associated with a set of independent parameters $\alpha_1$ and $\alpha_2$ (Fig. 1.1) such that

$$
x_1 = f_1(\alpha_1, \alpha_2), \quad x_2 = f_2(\alpha_1, \alpha_2), \quad x_3 = f_3(\alpha_1, \alpha_2),
$$

(1.1)

where $f_j (j = 1, 2, 3)$ are single-valued functions that possess derivatives up to any required order. By putting $\alpha_1 = \text{constant}$ and varying the parameter $\alpha_2$ in $f_j(c, \alpha_2)$, we obtain a curve that lies entirely on $S$. By successively giving $\alpha_1$ a series of constant values we obtain a family of curves along which only a parameter $\alpha_2$ varies. These curves are called the $\alpha_2$-coordinate lines. Similarly, on setting $\alpha_2 = \text{constant}$ we obtain the $\alpha_1$-coordinate lines of $S$. We assume that only one curve of the family passes through a point of the given surface. Thus, any point $M$ on $S$ can be treated as a cross-intersection of the $\alpha_1$ and $\alpha_2$ curvilinear coordinate lines.

The position of a point $M$ with respect to the origin $O$ of the reference system is defined by the position vector $\vec{r}$,

$$
\vec{r} = \vec{i}_1 x_1 + \vec{i}_2 x_2 + \vec{i}_3 x_3 = \sum_{i=1}^{3} \vec{i}_i x_i,
$$

where $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ is the orthonormal triad of unit vectors associated with $\{x_1, x_2, x_3\}$. By virtue of Eqs. (1.1) it can be written in the form

$$
\vec{r} = \vec{i}_1 f_1(\alpha_1, \alpha_2) + \vec{i}_2 f_2(\alpha_1, \alpha_2) + \vec{i}_3 f_3(\alpha_1, \alpha_2).
$$

Equation (1.2) is the vector equation of a surface. On differentiating $\vec{r}$ with respect to $\alpha_i (i = 1, 2)$ vectors tangent to the $\alpha_1$- and $\alpha_2$-coordinate lines are found to be

$$
\vec{r}_1 = \frac{\partial \vec{r}}{\partial \alpha_1}, \quad \vec{r}_2 = \frac{\partial \vec{r}}{\partial \alpha_2}.
$$

(1.3)
1.1 Intrinsic geometry

Fig. 1.1 Intrinsic parameterization of the surface.

Modules and the scalar product of $\vec{r}_i$ $(i = 1, 2)$ are defined by

\[ |\vec{r}_1| = \vec{r}_1 \cdot \vec{r}_1 = A_1 = \sqrt{a_{11}}, \quad |\vec{r}_2| = \vec{r}_2 \cdot \vec{r}_2 = A_2 = \sqrt{a_{22}}, \]

\[ \vec{r}_1 \cdot \vec{r}_2 = A_1 A_2 \cos \chi = \sqrt{a_{12}}, \tag{1.4} \]

where $\chi$ is the angle between coordinate lines, $A_i$ are the Lamé parameters and $a_{ik}$ are the coefficients of the metric tensor $\mathbf{A}$ on $S$. Using Eqs. (1.4) we introduce the unit vectors $\vec{e}_i$ in the direction of $\vec{r}_i$ which are described by

\[ \vec{e}_1 = \frac{\vec{r}_1}{|\vec{r}_1|} = \frac{\vec{r}_1}{A_1}, \quad \vec{e}_2 = \frac{\vec{r}_2}{|\vec{r}_2|} = \frac{\vec{r}_2}{A_2}. \tag{1.5} \]

The vector $\vec{m}$ normal to $\vec{r}_1$ and $\vec{r}_2$ is found from

\[ \vec{m} = \vec{r}_1 \times \vec{r}_2 \quad \text{and} \quad \vec{m} \cdot \vec{r}_1 = 0, \quad \vec{m} \cdot \vec{r}_2 = 0, \tag{1.6} \]

where $\vec{r}_1 \times \vec{r}_2$ is the vector product. The vectors $\vec{r}_1, \vec{r}_2$ and $\vec{m}$ are linearly independent and comprise a covariant base $\{\vec{r}_1, \vec{r}_2, \vec{m}\}$ on $S$. The reciprocal base $\{\vec{r}^1, \vec{r}^2, \vec{m}\}$ is defined by

\[ \vec{r}^1 = \frac{\vec{r}_2 \times \vec{m}}{\vec{r}_1 (\vec{r}_2 \times \vec{m})}, \quad \vec{r}^2 = \frac{\vec{m} \times \vec{r}_1}{\vec{r}_2 (\vec{m} \times \vec{r}_1)}, \tag{1.7} \]

where $\vec{r}_1 (\vec{m} \times \vec{r}_2)$ is the scalar triple product. Evidently, the vectors $\vec{r}^k$ and $\vec{r}_i$ are mutually orthogonal, i.e.

\[ \vec{r}^k \cdot \vec{r}_i = \delta^k_i, \quad \vec{r}^k \cdot \vec{m} = 0. \]

Here $\delta^k_i$ is the Kronecker delta such that $\delta^k_i = 1$ if $i = k$ and $\delta^k_i = 0$ if $i \neq k$.

Let $\vec{m} (\vec{r}_1 \times \vec{r}_2) = c_{ik}$ and $\vec{m} (\vec{r}^2 \times \vec{r}^k) = c^{ik}$. Hence,

\[ c_{ik} \vec{m} = \vec{r}_1 \times \vec{r}_k, \quad c^{ik} \vec{m} = \vec{r}^j \times \vec{r}^k, \]

\[ c_{ik} \vec{r}^k = \vec{m} \times \vec{r}_i, \quad c^{ik} \vec{r}_i = \vec{m} \times \vec{r}^j. \tag{1.8} \]
It follows that
\[ c_{ii} = 0, \quad c_{12} = -c_{12} = 1 / \sqrt{a}, \]
where \( a = (A_1 A_2 \sin \chi)^2 \), and
\[ c_{ik} c^{km} = \delta_i^m, \quad c_{ik} c^{ji} = \delta_j^i \quad (i, k = 1, 2). \]

The length of a line element between two infinitely close points \( M(x_1, x_2, x_3) \) and \( N(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) \) (Fig. 1.2) is given by
\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = |d\vec{x}|^2 = |\vec{r}_1 d\alpha_1|^2. \]

Using Eqs. (1.4) in the above, we have
\[ ds^2 = A_1^2 d\alpha_1^2 + 2A_1 A_2 \cos \chi d\alpha_1 d\alpha_2 + A_2^2 d\alpha_2^2 = a_{ik} d\alpha_i d\alpha_k. \]

The quadratic form (Eq. (1.11)) is called the first fundamental form of the surface. It allows us to calculate the length of line elements, the angle between coordinate curves and the surface area
\[ ds_A = |\vec{r}_1 \times \vec{r}_2| d\alpha_1 d\alpha_2 = \sqrt{a} d\alpha_1 d\alpha_2, \]
and therefore it fully describes the intrinsic geometry of \( S \).

**1.2 Extrinsic geometry**

Let \( \Gamma \) be a non-singular curve on \( S \) parameterized by arc length \( s \) (Fig. 1.3)
\[ \vec{r} = \vec{r}(s) = \vec{r}(\alpha_1(s), \alpha_2(s)). \]

By differentiating \( \vec{r}(s) \) with respect to \( s \) the unit vector \( \vec{r} \) tangent to \( \Gamma \) is found to be
1.2 Extrinsic geometry

By applying the Frenet–Serret formula for the derivative of $\tau$ with respect to $s$ we get

$$\frac{d\tau}{ds} = \tilde{\tau} = \tilde{r}_1 \frac{d\alpha_1}{ds} + \tilde{r}_2 \frac{d\alpha_2}{ds}. \quad (1.13)$$

By applying the Frenet–Serret formula for the derivative of $\tau$ with respect to $s$ we get

$$\frac{d\tilde{\tau}}{ds} = \tilde{n}, \quad (1.14)$$

where $1/R_c$ is the curvature and $\tilde{n}$ is the vector normal to $\Gamma$. By substituting Eq. (1.7) into (1.8) we obtain

$$\tilde{n} = \sum_{i=1}^{2} \sum_{k=1}^{2} \tilde{r}_{ik} \frac{d\alpha_i}{ds} \frac{d\alpha_k}{ds} + \tilde{r}_1 \frac{d\alpha_1}{ds} + \tilde{r}_2 \frac{d\alpha_2}{ds}, \quad (1.15)$$

where

$$\tilde{r}_{ik} = \frac{\partial^2 \tau}{\partial \alpha_i \partial \alpha_k}, \quad \tilde{r}_{ik} = \tilde{r}_{ki}.$$ 

Let $\phi$ be the angle between the vectors $\tilde{m}$ and $\tilde{n}$ such that $\tilde{m}\tilde{n} = \cos \phi$. Then the scalar product of Eq. (1.15) with $\tilde{m}$ yields

$$\frac{\cos \phi}{R_c} = b_{11} \frac{d\alpha_1^2}{ds^2} + 2b_{12} \frac{d\alpha_1}{ds} \frac{d\alpha_2}{ds} + b_{22} \frac{d\alpha_2^2}{ds^2}, \quad (1.16)$$

where

$$b_{11} = \tilde{m}\tilde{r}_{11}, \quad b_{12} = \tilde{m}\tilde{r}_{12} = \tilde{m}\tilde{r}_{21}, \quad b_{22} = \tilde{m}\tilde{r}_{22}. \quad (1.17)$$

The quadratic form

$$b_{11} \frac{d\alpha_1^2}{ds^2} + 2b_{12} \frac{d\alpha_1}{ds} \frac{d\alpha_2}{ds} + b_{22} \frac{d\alpha_2^2}{ds^2}$$

is called the second fundamental form of the surface. On differentiating Eq. (1.6) with respect to $\alpha_i$ we find

Fig. 1.3 The extrinsic geometry of the surface and a local base $\{\tilde{n}, \tilde{n}_b, \tilde{\tau}\}$ associated with a curve $\Gamma$. 

\[\tilde{\tau} = \frac{d\tilde{r}}{ds} = \tilde{r}_1 \frac{d\alpha_1}{ds} + \tilde{r}_2 \frac{d\alpha_2}{ds}. \quad (1.13)\]
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The geometry of the surface

\[ b_{ik} = -\bar{m}_{i} \bar{r}_{k} = -\bar{m}_{i} \bar{r}_{i}, \quad (1.18) \]

where

\[ \bar{m}_{i} = \frac{\partial \bar{m}}{\partial \alpha_{i}}. \quad (1.19) \]

A normal section at any \( M(\alpha_{1}, \alpha_{2}) \in S \) is the section by some plane that contains the vector \( \bar{m} \perp S \). Assuming \( \phi = \pi \), which implies that \( \bar{m} \) and \( \bar{n} \) are oriented in opposite directions, from Eq. (1.10) for the curvature of the normal section \( 1/R_{n} \), we obtain

\[ -\frac{1}{R_{n}} = \frac{b_{11} \, d\alpha_{1}^{2} + 2b_{12} \, d\alpha_{1} \, d\alpha_{2} + b_{22} \, d\alpha_{2}^{2}}{A_{1}^{2} \, d\alpha_{1}^{2} + 2A_{1}A_{2} \, d\alpha_{1} \, d\alpha_{2} + A_{2}^{2} \, d\alpha_{2}^{2}}. \quad (1.20) \]

Henceforth, we assume that the coordinate lines are arranged in such a way that \( \bar{m} \) is positive when pointing from the concave to the convex side of the surface. On putting \( \alpha_{2} = \text{constant} \) and \( \alpha_{1} = \text{constant} \) in Eq. (1.20), for the curvatures \( k_{11} \) and \( k_{22} \) of the normal sections in the directions of \( \alpha_{1} \) and \( \alpha_{2} \) we find

\[ \frac{1}{R_{\alpha_{1}}} := k_{11} = -\frac{b_{11}}{A_{1}^{2}}, \quad \frac{1}{R_{\alpha_{2}}} := k_{22} = -\frac{b_{22}}{A_{2}^{2}}, \quad (1.21a) \]

and the twist \( k_{12} \) of the surface

\[ \frac{1}{R_{\alpha_{1}\alpha_{2}}} := k_{12} = -\frac{b_{12}}{A_{1}A_{2}}. \quad (1.21b) \]

It becomes evident from the above considerations that the second fundamental form describes the intrinsic geometry of the surface.

At any point \( M(\alpha_{1}, \alpha_{2}) \in S \) there exist two normal sections where \( 1/R_{n} \) assumes extreme values. They are called principal sections. The two perpendicular directions at \( M \) belonging to the corresponding tangent plane are called the principal directions and the principal curvatures are \( (1/R)_{\text{max}} = 1/R_{1} \) and \( (1/R)_{\text{min}} = 1/R_{2} \) (Fig. 1.4). Thus, there is at least one set of principal directions at any point on \( S \). A curve on the surface such that the tangent at any point to it is collinear with the principal direction is called the line of curvature. Thus, two lines of curvature intersect at right angles and pass through each point of \( S \). We assume that the coordinate lines \( \alpha_{1} \) and \( \alpha_{2} \) are the lines of curvature \( (\chi = \pi/2, b_{12} = 0) \). Such coordinates have an advantage over other coordinate systems since the governing equations in them have a relatively simple form.

Let \( \bar{r}_{ik} \) be second derivatives of the position vector with respect to \( \alpha_{i}(k) \) \( (i, k = 1, 2) \). On decomposing \( \bar{r}_{ik} \) with respect to the covariant base \( \{\bar{r}_{1}, \bar{r}_{2}, \bar{m}\} \) we get

\[ \bar{r}_{ik} = \Gamma_{ik}^{1} \bar{r}_{1} + \Gamma_{ik}^{2} \bar{r}_{2} + \bar{m} b_{ik}. \quad (1.22) \]