

## AXIOMATIC THEORIES OF TRUTH

At the centre of the traditional discussion of truth is the question of how truth is defined. Recent research, especially with the development of deflationist accounts of truth, has tended to take truth as an undefined primitive notion governed by axioms, while the liar paradox and cognate paradoxes pose problems for certain seemingly natural axioms for truth. In this book, Volker Halbach examines the most important axiomatizations of truth, explores their properties, and shows how the logical results impinge on the philosophical topics related to truth. For instance, he shows how the discussion of topics such as deflationism depends on the solution of the paradoxes. His book is an invaluable survey of the logical background to the philosophical discussion of truth, and will be indispensable reading for any graduate and professional philosopher in theories of truth.

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## Preface

This book has four parts. In the first part I sketch some mathematical preliminaries, fix notational conventions, and outline some motivations for studying axiomatic theories of truth. Deeper philosophical investigation, however, is postponed to the last part when the significance of the formal results is discussed. The axiomatic theories of truth and the results about them are then given in the two central parts. The first of them is devoted to typed theories, that is, to theories where the truth predicate applies provably only to sentences not containing the truth predicate. In the third part of the book I discuss type-free theories of truth and how inconsistency can be avoided without Tarski's object and metalanguage distinction. In the fourth and final part, the philosophical implications of the formal results are evaluated.

I have tried to make the book usable as a handbook of axiomatic truth theories, so that one can dip into various sections without having read all the preceding material. To this end I have also included many cross references and occasionally repeated some explanations concerning notation. It should be possible to read the final part on philosophical issues without having read the two formal parts containing the formal results. However, this last part presupposes some familiarity with the notation introduced in Chapters 5 and 6 in the first part. Of course, when discussing philosophical issues I will refer back to the formal results obtained in the two previous parts, and the reader who is interested in the last part only and skips the two formal parts will have to take my word for them.

All parts of the book should be accessible to a reader who has some acquaintance with the proofs of the Gödel incompleteness theorems and therefore with the basic concepts of recursion theory and metamathematics. In sections where I have used techniques from other areas of logic like model theory, I have defined all notions I use and have made most proofs so explicit that they should be accessible to readers not used to these techniques. I also assume very little with respect to proof theory: I do not use methods from ordinal analysis as I felt unable to provide an introduction to this branch of proof theory in a short chapter. In many research papers axiomatic truth theories are analysed by relating them to well-investigated subsystems

of second-order number theory. The truth theories can then be compared via well-known results about these subsystems. Here I have attempted, whenever possible, to relate the theories of truth to one another directly, without the detour via the second-order systems, obviating the need to appeal to well-documented or folklore results about them.

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