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978-0-521-11470-7 - Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century

Ranjan Roy

Table of Contents

[More information](#)

## *Contents*

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<i>Preface</i>	<i>page xvii</i>
1 Power Series in Fifteenth-Century Kerala	1
1.1 Preliminary Remarks	1
1.2 Transformation of Series	4
1.3 Jyesthadeva on Sums of Powers	5
1.4 Arctangent Series in the <i>Yuktibhasa</i>	7
1.5 Derivation of the Sine Series in the <i>Yuktibhasa</i>	8
1.6 Continued Fractions	10
1.7 Exercises	12
1.8 Notes on the Literature	14
2 Sums of Powers of Integers	16
2.1 Preliminary Remarks	16
2.2 Johann Faulhaber and Sums of Powers	19
2.3 Jakob Bernoulli's Polynomials	20
2.4 Proof of Bernoulli's Formula	24
2.5 Exercises	25
2.6 Notes on the Literature	26
3 Infinite Product of Wallis	28
3.1 Preliminary Remarks	28
3.2 Wallis's Infinite Product for $\pi$	32
3.3 Brouncker and Infinite Continued Fractions	33
3.4 Stieltjes: Probability Integral	36
3.5 Euler: Series and Continued Fractions	38
3.6 Euler: Products and Continued Fractions	40
3.7 Euler: Continued Fractions and Integrals	43
3.8 Sylvester: A Difference Equation and Euler's Continued Fraction	45
3.9 Euler: Riccati's Equation and Continued Fractions	46
3.10 Exercises	48
3.11 Notes on the Literature	50

vi	<i>Contents</i>	
4	The Binomial Theorem	51
4.1	Preliminary Remarks	51
4.2	Landen's Derivation of the Binomial Theorem	57
4.3	Euler's Proof for Rational Indices	58
4.4	Cauchy: Proof of the Binomial Theorem for Real Exponents	60
4.5	Abel's Theorem on Continuity	62
4.6	Harkness and Morley's Proof of the Binomial Theorem	66
4.7	Exercises	67
4.8	Notes on the Literature	69
5	The Rectification of Curves	71
5.1	Preliminary Remarks	71
5.2	Descartes's Method of Finding the Normal	73
5.3	Hudde's Rule for a Double Root	74
5.4	Van Heuraet's Letter on Rectification	75
5.5	Newton's Rectification of a Curve	76
5.6	Leibniz's Derivation of the Arc Length	77
5.7	Exercises	78
5.8	Notes on the Literature	79
6	Inequalities	81
6.1	Preliminary Remarks	81
6.2	Harriot's Proof of the Arithmetic and Geometric Means Inequality	87
6.3	Maclaurin's Inequalities	88
6.4	Jensen's Inequality	89
6.5	Reisz's Proof of Minkowski's Inequality	90
6.6	Exercises	91
6.7	Notes on the Literature	96
7	Geometric Calculus	97
7.1	Preliminary Remarks	97
7.2	Pascal's Evaluation of $\int \sin x \, dx$	100
7.3	Gregory's Evaluation of a Beta Integral	101
7.4	Gregory's Evaluation of $\int \sec \theta \, d\theta$	104
7.5	Barrow's Evaluation of $\int \sec \theta \, d\theta$	106
7.6	Barrow and the Integral $\int \sqrt{x^2 + a^2} \, dx$	108
7.7	Barrow's Proof of $\frac{d}{d\theta} \tan \theta = \sec^2 \theta$	110
7.8	Barrow's Product Rule for Derivatives	111
7.9	Barrow's Fundamental Theorem of Calculus	114
7.10	Exercises	114
7.11	Notes on the Literature	118
8	The Calculus of Newton and Leibniz	120
8.1	Preliminary Remarks	120
8.2	Newton's 1671 Calculus Text	123
8.3	Leibniz: Differential Calculus	126

Cambridge University Press

978-0-521-11470-7 - Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century

Ranjan Roy

Table of Contents

[More information](#)

<i>Contents</i>		vii
8.4	Leibniz on the Catenary	129
8.5	Johann Bernoulli on the Catenary	131
8.6	Johann Bernoulli: The Brachistochrone	132
8.7	Newton's Solution to the Brachistochrone	133
8.8	Newton on the Radius of Curvature	135
8.9	Johann Bernoulli on the Radius of Curvature	136
8.10	Exercises	137
8.11	Notes on the Literature	138
9	De Analysi per Aequationes Infinitas	140
9.1	Preliminary Remarks	140
9.2	Algebra of Infinite Series	142
9.3	Newton's Polygon	145
9.4	Newton on Differential Equations	146
9.5	Newton's Earliest Work on Series	147
9.6	De Moivre on Newton's Formula for $\sin n\theta$	149
9.7	Stirling's Proof of Newton's Formula	150
9.8	Zolotarev: Lagrange Inversion with Remainder	152
9.9	Exercises	153
9.10	Notes on the Literature	156
10	Finite Differences: Interpolation and Quadrature	158
10.1	Preliminary Remarks	158
10.2	Newton: Divided Difference Interpolation	163
10.3	Gregory–Newton Interpolation Formula	165
10.4	Waring, Lagrange: Interpolation Formula	165
10.5	Cauchy, Jacobi: Lagrange Interpolation Formula	166
10.6	Newton on Approximate Quadrature	168
10.7	Hermite: Approximate Integration	170
10.8	Chebyshev on Numerical Integration	172
10.9	Exercises	173
10.10	Notes on the Literature	175
11	Series Transformation by Finite Differences	176
11.1	Preliminary Remarks	176
11.2	Newton's Transformation	181
11.3	Montmort's Transformation	182
11.4	Euler's Transformation Formula	184
11.5	Stirling's Transformation Formulas	187
11.6	Nicole's Examples of Sums	190
11.7	Stirling Numbers	191
11.8	Lagrange's Proof of Wilson's Theorem	194
11.9	Taylor's Summation by Parts	195
11.10	Exercises	196
11.11	Notes on the Literature	199

viii	<i>Contents</i>	
12	The Taylor Series	200
	12.1 Preliminary Remarks	200
	12.2 Gregory's Discovery of the Taylor Series	206
	12.3 Newton: An Iterated Integral as a Single Integral	209
	12.4 Bernoulli and Leibniz: A Form of the Taylor Series	210
	12.5 Taylor and Euler on the Taylor Series	211
	12.6 Lacroix on d'Alembert's Derivation of the Remainder	212
	12.7 Lagrange's Derivation of the Remainder Term	213
	12.8 Laplace's Derivation of the Remainder Term	215
	12.9 Cauchy on Taylor's Formula and l'Hôpital's Rule	216
	12.10 Cauchy: The Intermediate Value Theorem	218
	12.11 Exercises	219
	12.12 Notes on the Literature	220
13	Integration of Rational Functions	222
	13.1 Preliminary Remarks	222
	13.2 Newton's 1666 Basic Integrals	228
	13.3 Newton's Factorization of $x^n \pm 1$	230
	13.4 Cotes and de Moivre's Factorizations	231
	13.5 Euler: Integration of Rational Functions	233
	13.6 Euler's Generalization of His Earlier Work	234
	13.7 Hermite's Rational Part Algorithm	237
	13.8 Johann Bernoulli: Integration of $\sqrt{ax^2 + bx + c}$	238
	13.9 Exercises	239
	13.10 Notes on the Literature	243
14	Difference Equations	245
	14.1 Preliminary Remarks	245
	14.2 De Moivre on Recurrent Series	247
	14.3 Stirling's Method of Ultimate Relations	250
	14.4 Daniel Bernoulli on Difference Equations	252
	14.5 Lagrange: Nonhomogeneous Equations	254
	14.6 Laplace: Nonhomogeneous Equations	257
	14.7 Exercises	258
	14.8 Notes on the Literature	259
15	Differential Equations	260
	15.1 Preliminary Remarks	260
	15.2 Leibniz: Equations and Series	268
	15.3 Newton on Separation of Variables	270
	15.4 Johann Bernoulli's Solution of a First-Order Equation	271
	15.5 Euler on General Linear Equations with Constant Coefficients	272
	15.6 Euler: Nonhomogeneous Equations	274
	15.7 Lagrange's Use of the Adjoint	276
	15.8 Jakob Bernoulli and Riccati's Equation	278
	15.9 Riccati's Equation	278

<i>Contents</i>		ix
15.10	Singular Solutions	279
15.11	Mukhopadhyay on Monge's Equation	283
15.12	Exercises	285
15.13	Notes on the Literature	287
16	Series and Products for Elementary Functions	289
16.1	Preliminary Remarks	289
16.2	Euler: Series for Elementary Functions	292
16.3	Euler: Products for Trigonometric Functions	293
16.4	Euler's Finite Product for $\sin nx$	294
16.5	Cauchy's Derivation of the Product Formulas	295
16.6	Euler and Niklaus I Bernoulli: Partial Fractions Expansions of Trigonometric Functions	298
16.7	Euler: Dilogarithm	301
16.8	Landen's Evaluation of $\zeta(2)$	302
16.9	Spence: Two-Variable Dilogarithm Formula	304
16.10	Exercises	306
16.11	Notes on the Literature	310
17	Solution of Equations by Radicals	311
17.1	Preliminary Remarks	311
17.2	Viète's Trigonometric Solution of the Cubic	316
17.3	Descartes's Solution of the Quartic	318
17.4	Euler's Solution of a Quartic	319
17.5	Gauss: Cyclotomy, Lagrange Resolvents, and Gauss Sums	320
17.6	Kronecker: Irreducibility of the Cyclotomic Polynomial	324
17.7	Exercises	325
17.8	Notes on the Literature	325
18	Symmetric Functions	326
18.1	Preliminary Remarks	326
18.2	Euler's Proofs of Newton's Rule	331
18.3	Maclaurin's Proof of Newton's Rule	332
18.4	Waring's Power Sum Formula	334
18.5	Gauss's Fundamental Theorem of Symmetric Functions	334
18.6	Cauchy: Fundamental Theorem of Symmetric Functions	335
18.7	Cauchy: Elementary Symmetric Functions as Rational Functions of Odd Power Sums	336
18.8	Laguerre and Pólya on Symmetric Functions	337
18.9	MacMahon's Generalization of Waring's Formula	340
18.10	Exercises	343
18.11	Notes on the Literature	344
19	Calculus of Several Variables	346
19.1	Preliminary Remarks	346
19.2	Homogeneous Functions	351
19.3	Cauchy: Taylor Series in Several Variables	352

Cambridge University Press

978-0-521-11470-7 - Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century

Ranjan Roy

Table of Contents

[More information](#)

x	<i>Contents</i>	
	19.4 Clairaut: Exact Differentials and Line Integrals	354
	19.5 Euler: Double Integrals	356
	19.6 Lagrange's Change of Variables Formula	358
	19.7 Green's Integral Identities	359
	19.8 Riemann's Proof of Green's Formula	361
	19.9 Stokes's Theorem	362
	19.10 Exercises	365
	19.11 Notes on the Literature	365
20	Algebraic Analysis: The Calculus of Operations	367
	20.1 Preliminary Remarks	367
	20.2 Lagrange's Extension of the Euler–Maclaurin Formula	375
	20.3 Français's Method of Solving Differential Equations	379
	20.4 Herschel: Calculus of Finite Differences	380
	20.5 Murphy's Theory of Analytical Operations	382
	20.6 Duncan Gregory's Operational Calculus	384
	20.7 Boole's Operational Calculus	387
	20.8 Jacobi and the Symbolic Method	390
	20.9 Cartier: Gregory's Proof of Leibniz's Rule	392
	20.10 Hamilton's Algebra of Complex Numbers and Quaternions	393
	20.11 Exercises	397
	20.12 Notes on the Literature	398
21	Fourier Series	400
	21.1 Preliminary Remarks	400
	21.2 Euler: Trigonometric Expansion of a Function	406
	21.3 Lagrange on the Longitudinal Motion of the Loaded Elastic String	407
	21.4 Euler on Fourier Series	410
	21.5 Fourier: Linear Equations in Infinitely Many Unknowns	412
	21.6 Dirichlet's Proof of Fourier's Theorem	417
	21.7 Dirichlet: On the Evaluation of Gauss Sums	421
	21.8 Exercises	424
	21.9 Notes on the Literature	425
22	Trigonometric Series after 1830	427
	22.1 Preliminary Remarks	427
	22.2 The Riemann Integral	429
	22.3 Smith: Revision of Riemann and Discovery of the Cantor Set	431
	22.4 Riemann's Theorems on Trigonometric Series	432
	22.5 The Riemann–Lebesgue Lemma	436
	22.6 Schwarz's Lemma on Generalized Derivatives	436
	22.7 Cantor's Uniqueness Theorem	437
	22.8 Exercises	439
	22.9 Notes on the Literature	443

*Contents*

xi

23	The Gamma Function	444
23.1	Preliminary Remarks	444
23.2	Stirling: $\Gamma(1/2)$ by Newton–Bessel Interpolation	450
23.3	Euler’s Evaluation of the Beta Integral	453
23.4	Gauss’s Theory of the Gamma Function	457
23.5	Poisson, Jacobi, and Dirichlet: Beta Integrals	460
23.6	Bohr, Mollerup, and Artin on the Gamma Function	462
23.7	Kummer’s Fourier Series for $\ln \Gamma(x)$	465
23.8	Exercises	467
23.9	Notes on the Literature	474
24	The Asymptotic Series for $\ln \Gamma(x)$	476
24.1	Preliminary Remarks	476
24.2	De Moivre’s Asymptotic Series	481
24.3	Stirling’s Asymptotic Series	483
24.4	Binet’s Integrals for $\ln \Gamma(x)$	486
24.5	Cauchy’s Proof of the Asymptotic Character of de Moivre’s Series	488
24.6	Exercises	489
24.7	Notes on the Literature	493
25	The Euler–Maclaurin Summation Formula	494
25.1	Preliminary Remarks	494
25.2	Euler on the Euler–Maclaurin Formula	499
25.3	Maclaurin’s Derivation of the Euler–Maclaurin Formula	501
25.4	Poisson’s Remainder Term	503
25.5	Jacobi’s Remainder Term	505
25.6	Euler on the Fourier Expansions of Bernoulli Polynomials	507
25.7	Abel’s Derivation of the Planar–Abel Formula	508
25.8	Exercises	509
25.9	Notes on the Literature	513
26	$L$ -Series	515
26.1	Preliminary Remarks	515
26.2	Euler’s First Evaluation of $\sum 1/n^{2k}$	521
26.3	Euler: Bernoulli Numbers and $\sum 1/n^{2k}$	522
26.4	Euler’s Evaluation of Some $L$ -Series Values by Partial Fractions	524
26.5	Euler’s Evaluation of $\sum 1/n^2$ by Integration	525
26.6	N. Bernoulli’s Evaluation of $\sum 1/(2n+1)^2$	527
26.7	Euler and Goldbach: Double Zeta Values	528
26.8	Dirichlet’s Summation of $L(1, \chi)$	532
26.9	Eisenstein’s Proof of the Functional Equation	535
26.10	Riemann’s Derivations of the Functional Equation	536
26.11	Euler’s Product for $\sum 1/n^s$	539
26.12	Dirichlet Characters	540
26.13	Exercises	542
26.14	Notes on the Literature	545

xii	<i>Contents</i>	
27	The Hypergeometric Series	547
	27.1 Preliminary Remarks	547
	27.2 Euler's Derivation of the Hypergeometric Equation	555
	27.3 Pfaff's Derivation of the ${}_3F_2$ Identity	556
	27.4 Gauss's Contiguous Relations and Summation Formula	557
	27.5 Gauss's Proof of the Convergence of $F(a, b, c, x)$ for $c - a - b > 0$	559
	27.6 Gauss's Continued Fraction	560
	27.7 Gauss: Transformations of Hypergeometric Functions	561
	27.8 Kummer's 1836 Paper on Hypergeometric Series	564
	27.9 Jacobi's Solution by Definite Integrals	565
	27.10 Riemann's Theory of Hypergeometric Functions	567
	27.11 Exercises	569
	27.12 Notes on the Literature	572
28	Orthogonal Polynomials	574
	28.1 Preliminary Remarks	574
	28.2 Legendre's Proof of the Orthogonality of His Polynomials	578
	28.3 Gauss on Numerical Integration	579
	28.4 Jacobi's Commentary on Gauss	582
	28.5 Murphy and Ivory: The Rodrigues Formula	583
	28.6 Liouville's Proof of the Rodrigues Formula	585
	28.7 The Jacobi Polynomials	587
	28.8 Chebyshev: Discrete Orthogonal Polynomials	590
	28.9 Chebyshev and Orthogonal Matrices	594
	28.10 Chebyshev's Discrete Legendre and Jacobi Polynomials	594
	28.11 Exercises	596
	28.12 Notes on the Literature	597
29	$q$ -Series	599
	29.1 Preliminary Remarks	599
	29.2 Jakob Bernoulli's Theta Series	605
	29.3 Euler's $q$ -series Identities	605
	29.4 Euler's Pentagonal Number Theorem	606
	29.5 Gauss: Triangular and Square Numbers Theorem	608
	29.6 Gauss Polynomials and Gauss Sums	611
	29.7 Gauss's $q$ -Binomial Theorem and the Triple Product Identity	615
	29.8 Jacobi: Triple Product Identity	617
	29.9 Eisenstein: $q$ -Binomial Theorem	618
	29.10 Jacobi's $q$ -Series Identity	619
	29.11 Cauchy and Ramanujan: The Extension of the Triple Product	621
	29.12 Rodrigues and MacMahon: Combinatorics	622
	29.13 Exercises	623
	29.14 Notes on the Literature	625



*Contents*

xiii

30	Partitions	627
	30.1 Preliminary Remarks	627
	30.2 Sylvester on Partitions	638
	30.3 Cayley: Sylvester's Formula	642
	30.4 Ramanujan: Rogers–Ramanujan Identities	644
	30.5 Ramanujan's Congruence Properties of Partitions	646
	30.6 Exercises	649
	30.7 Notes on the Literature	651
31	$q$ -Series and $q$ -Orthogonal Polynomials	653
	31.1 Preliminary Remarks	653
	31.2 Heine's Transformation	661
	31.3 Rogers: Threefold Symmetry	662
	31.4 Rogers: Rogers–Ramanujan Identities	665
	31.5 Rogers: Third Memoir	670
	31.6 Rogers–Szegő Polynomials	671
	31.7 Feldheim and Lanzewizky: Orthogonality of $q$ -Ultraspherical Polynomials	673
	31.8 Exercises	677
	31.9 Notes on the Literature	679
32	Primes in Arithmetic Progressions	680
	32.1 Preliminary Remarks	680
	32.2 Euler: Sum of Prime Reciprocals	682
	32.3 Dirichlet: Infinitude of Primes in an Arithmetic Progression	683
	32.4 Class Number and $L_\chi(1)$	686
	32.5 De la Vallée Poussin's Complex Analytic Proof of $L_\chi(1) \neq 0$	688
	32.6 Gelfond and Linnik: Proof of $L_\chi(1) \neq 0$	689
	32.7 Monsky's Proof That $L_\chi(1) \neq 0$	691
	32.8 Exercises	692
	32.9 Notes on the Literature	694
33	Distribution of Primes: Early Results	695
	33.1 Preliminary Remarks	695
	33.2 Chebyshev on Legendre's Formula	701
	33.3 Chebyshev's Proof of Bertrand's Conjecture	705
	33.4 De Polignac's Evaluation of $\sum_{p \leq x} \frac{\ln p}{p}$	710
	33.5 Mertens's Evaluation of $\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}$	710
	33.6 Riemann's Formula for $\pi(x)$	714
	33.7 Exercises	717
	33.8 Notes on the Literature	719
34	Invariant Theory: Cayley and Sylvester	720
	34.1 Preliminary Remarks	720
	34.2 Boole's Derivation of an Invariant	729

34.3	Differential Operators of Cayley and Sylvester	733
34.4	Cayley's Generating Function for the Number of Invariants	736
34.5	Sylvester's Fundamental Theorem of Invariant Theory	740
34.6	Hilbert's Finite Basis Theorem	743
34.7	Hilbert's Nullstellensatz	746
34.8	Exercises	746
34.9	Notes on the Literature	747
35	Summability	749
35.1	Preliminary Remarks	749
35.2	Fejér: Summability of Fourier Series	760
35.3	Karamata's Proof of the Hardy–Littlewood Theorem	763
35.4	Wiener's Proof of Littlewood's Theorem	764
35.5	Hardy and Littlewood: The Prime Number Theorem	766
35.6	Wiener's Proof of the PNT	768
35.7	Kac's Proof of Wiener's Theorem	771
35.8	Gelfand: Normed Rings	772
35.9	Exercises	775
35.10	Notes on the Literature	777
36	Elliptic Functions: Eighteenth Century	778
36.1	Preliminary Remarks	778
36.2	Fagnano Divides the Lemniscate	786
36.3	Euler: Addition Formula	790
36.4	Cayley on Landen's Transformation	791
36.5	Lagrange, Gauss, Ivory on the agM	794
36.6	Remarks on Gauss and Elliptic Functions	800
36.7	Exercises	811
36.8	Notes on the Literature	813
37	Elliptic Functions: Nineteenth Century	816
37.1	Preliminary Remarks	816
37.2	Abel: Elliptic Functions	821
37.3	Abel: Infinite Products	823
37.4	Abel: Division of Elliptic Functions and Algebraic Equations	826
37.5	Abel: Division of the Lemniscate	830
37.6	Jacobi's Elliptic Functions	832
37.7	Jacobi: Cubic and Quintic Transformations	834
37.8	Jacobi's Transcendental Theory of Transformations	839
37.9	Jacobi: Infinite Products for Elliptic Functions	844
37.10	Jacobi: Sums of Squares	847
37.11	Cauchy: Theta Transformations and Gauss Sums	849
37.12	Eisenstein: Reciprocity Laws	852
37.13	Liouville's Theory of Elliptic Functions	858
37.14	Exercises	863
37.15	Notes on the Literature	865

Cambridge University Press

978-0-521-11470-7 - Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century

Ranjan Roy

Table of Contents

[More information](#)*Contents*

xv

38	Irrational and Transcendental Numbers	867
38.1	Preliminary Remarks	867
38.2	Liouville Numbers	878
38.3	Hermite's Proof of the Transcendence of $e$	880
38.4	Hilbert's Proof of the Transcendence of $e$	884
38.5	Exercises	885
38.6	Notes on the Literature	886
39	Value Distribution Theory	887
39.1	Preliminary Remarks	887
39.2	Jacobi on Jensen's Formula	892
39.3	Jensen's Proof	894
39.4	Bäcklund Proof of Jensen's Formula	895
39.5	R. Nevanlinna's Proof of the Poisson–Jensen Formula	896
39.6	Nevanlinna's First Fundamental Theorem	898
39.7	Nevanlinna's Factorization of a Meromorphic Function	901
39.8	Picard's Theorem	902
39.9	Borel's Theorem	902
39.10	Nevanlinna's Second Fundamental Theorem	903
39.11	Exercises	905
39.12	Notes on the Literature	906
40	Univalent Functions	907
40.1	Preliminary Remarks	907
40.2	Gronwall: Area Inequalities	914
40.3	Bieberbach's Conjecture	916
40.4	Littlewood: $ a_n  \leq en$	917
40.5	Littlewood and Paley on Odd Univalent Functions	918
40.6	Karl Löwner and the Parametric Method	920
40.7	De Branges: Proof of Bieberbach's Conjecture	923
40.8	Exercises	927
40.9	Notes on the Literature	928
41	Finite Fields	929
41.1	Preliminary Remarks	929
41.2	Euler's Proof of Fermat's Little Theorem	932
41.3	Gauss's Proof that $\mathbb{Z}_p^\times$ Is Cyclic	932
41.4	Gauss on Irreducible Polynomials Modulo a Prime	933
41.5	Galois on Finite Fields	936
41.6	Dedekind's Formula	939
41.7	Exercises	940
41.8	Notes on the Literature	941
	<i>References</i>	943
	<i>Index</i>	959