Sources in the Development of Mathematics

The discovery of infinite products by Wallis and infinite series by Newton marked the beginning of the modern mathematical era. The use of series allowed Newton to find the area under a curve defined by any algebraic equation, an achievement completely beyond the earlier methods of Torricelli, Fermat, and Pascal. The work of Newton and his contemporaries, including Leibniz and the Bernoullis, was concentrated in mathematical analysis and physics. Euler's prodigious mathematical accomplishments dramatically extended the scope of series and products to algebra, combinatorics, and number theory. Series and products proved pivotal in the work of Gauss, Abel, and Jacobi in elliptic functions; in Boole and Lagrange's operator calculus; and in Cayley, Sylvester, and Hilbert's invariant theory. Series and products still play a critical role in the mathematics of today. Consider the conjectures of Langlands, including that of Shimura-Taniyama, leading to Wiles's proof of Fermat's last theorem.

Drawing on the original work of mathematicians from Europe, Asia, and America, Ranjan Roy discusses many facets of the discovery and use of infinite series and products. He gives context and motivation for these discoveries, including original notation and diagrams when practical. He presents multiple derivations for many important theorems and formulas and provides interesting exercises, supplementing the results of each chapter.

Roy deals with numerous results, theorems, and methods used by students, mathematicians, engineers, and physicists. Moreover, since he presents original mathematical insights often omitted from textbooks, his work may be very helpful to mathematics teachers and researchers.

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Infinite Series and Products from the Fifteenth to the Twenty-first Century

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Preface

But this is something very important; one can render our youthful students no greater service than to give them suitable guidance, so that the advances in science become known to them through a study of the sources. – Weierstrass to Casorati, December 21, 1868

The development of infinite series and products marked the beginning of the modern mathematical era. In his *Arithmetica Infinitorum* of 1656, Wallis made groundbreaking discoveries in the use of such products and continued fractions. This work had a tremendous catalytic effect on the young Newton, leading him to the discovery of the binomial theorem for noninteger exponents. Newton explained in his *De Methodis* that the central pillar of his work in algebra and calculus was the powerful new method of infinite series. In letters written in 1670, James Gregory presented his discovery of several infinite series, most probably by means of finite difference interpolation formulas. Illustrating the very significant connections between series and finite difference methods, in the 1670s Newton made use of such methods to transform slowly convergent or even divergent series into rapidly convergent series, though he did not publish his results. Illustrating the importance of this approach, Montmort and Euler soon used new arguments to rediscover it. Newton further wrote in the *De Methodis* that he conceived of infinite series as analogues of infinite decimals, so that the four arithmetical operations and root extraction could be carried over to apply to variables. Thus, he applied infinite series to discover the inverse function and implicit function theorems. Newton concentrated largely on analysis and mathematical physics; Euler's prodigious intellect broadened Newton's conception to apply infinite series and products to number theory, algebra, and combinatorics; this legacy continues unabated even today.

Infinite series have numerous manifestations, including power series, trigonometric series, q-series, and Dirichlet series. Their scope and power are evident in their pivotal role in many areas of mathematics, including algebra, analysis, combinatorics, and number theory. As such, infinite series and products provide access to many mathematical questions and insights. For example, Maclaurin, Euler, and MacMahon studied symmetric functions using infinite series; Euler, Dirichlet, Chebyshev, and Riemann employed products and series to get deep insight into the distribution of primes. Gauss employed q-series to prove the law of quadratic reciprocity and Jacobi applied the triple product identity, also discovered by Gauss, to determine the number of representations of integers as sums of squares. Moreover, the correspondence between Daniel Bernoulli and Goldbach in the 1720s introduced the problem of determining whether a given series of rational numbers was irrational or transcendental. The 1843 publication of their letters prompted Liouville to lay the foundations of the theory of transcendental numbers.

The detailed table of contents at the beginning of this book may prove even more useful than the index in locating particular topics or questions. The preliminary remarks in each chapter provide some background on the origins and motivations of the ideas discussed in the subsequent, more detailed, and substantial sections of the chapter. The exercises following these sections offer references so that the reader may perhaps consult the original sources with a specific focus in mind. Most works cited in the notes at the end of each chapter should be readily accessible, especially since the number of books and papers online is increasing steadily.

Mathematics teachers and students may discover that the old sources, such as Simpson's books on algebra and calculus, Euler's *Introductio*, or the correspondence of Euler and Goldbach and the Bernoullis, are fruitful resources for calculus projects or undergraduate or graduate seminar topics. Since early mathematicians often omitted to mention the conditions under which their results would hold, analysis students could find it very instructive to work out the range of validity of those results. For example, Landen's formula for the dilogarithm, while very insightful and significant, is incorrect for a range of values, even where the series converge. At an advanced level, important research has arisen out of a study of old works. Indeed, by studying Descartes and Newton, Laguerre revived a subject others had abandoned for two hundred years and did his excellent work in numerical solutions of algebraic equations and extensions of the rule of signs. Again, André Weil recounted in his 1972 Ritt lectures on number theory that he arrived at the Weil conjectures through a study of Gauss's two papers on biquadratic residues.

It is edifying and a lot of fun to read the noteworthy works of long ago; this is common practice in literature and is equally appropriate and beneficial in mathematics. For example, a calculus student might enjoy and learn from Cotes's 1714 paper on logarithms or Maria Agnesi's 1748 treatment of the same topic in her work on analysis. At a more advanced level, Euler gave not just one or two but at least eight derivations of his famous formula $\sum 1/n^2 = \pi^2/6$. Reading these may serve to enlighten us on the variety of approaches to the perennial problem of summing series, though most of these approaches are not mentioned in textbooks. Students of literature routinely learn from and enjoy reading the words of, say, Austen, Hawthorne, Turgenev, or Shakespeare. We may likewise deepen our understanding and enjoyment of mathematics by reading and rereading the original works of mathematicians such as Barrow, Laplace, Chebyshev, or Newton. It might prove rewarding if mathematicians and students of mathematics were to make such reading a regular practice. In the introduction to his *Development of Mathematics in the 19th Century*, Felix Klein wrote, "Thus, it is impossible to grasp even *one* mathematical concept without having assimilated all the concepts which led up to its creation, and their connections."

Wherever practical, I have tried to present a mathematician's own notational methods. Seeing an argument in its original form is often instructive and can give us insight into its motivations and underlying rationale. Because of the numerous notations for logarithms, for simplicity I have denoted the logarithm of a real value by the familiar ln; in the case of complex or non-e-based logarithms, I have used log.

I am indebted to many persons who helped me in writing this book. I would first like to thank my wife, Gretchen Roy, for her invaluable assistance in editing and preparing the manuscript. I am grateful to Kalyan for his beautiful cover art. I thank my colleagues: Paul Campbell for his expert and generous assistance with the indexes and typesetting and Bruce Atwood for so cheerfully and accurately preparing the figures as they now appear. I am grateful to Ashish Thapa for his skillful typesetting and figure construction. Many thanks to Doreen Dalman, who typeset the majority of the book and did valuable troubleshooting. I am obliged to Paul Campbell and David Heesen for their meticulous work on the bibliography. I benefited from the input of those who read preliminary drafts of some chapters: Richard Askey, George Andrews, Lonnie Fairchild, Atar Mittal, Yu Shun, and Phil Straffin. I was fortunate to receive assistance from very capable librarians: Cindy Cooley and Chris Nelson at Beloit College, Travis Warwick at the Kleene Mathematics Library at the University of Wisconsin, the efficient librarians at the University Library in Cambridge, and the kind librarians at St. Andrews University Library. A. W. F. Edwards, of Gonville and Caius College, also gave me helpful guidance. I am grateful for financial and other assistance from Beloit College; thanks to John Burris, Lynn Franken, and Ann Davies for their encouragement and support. Heartfelt gratitude goes to Maitreyi Lagunas, Margaret Carey, Mihir Banerjee, Sahib Ram Mandan, and Ramendra Bose. Finally, I am deeply indebted to my parents for their intellectual, emotional, and practical support of my efforts to become a mathematician. I dedicate this book to their memory.

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