1

Power Series in Fifteenth-Century Kerala

1.1 Preliminary Remarks

The Indian astronomer and mathematician Madhava (c. 1340–c. 1425) discovered infinite power series about two and a half centuries before Newton rediscovered them in the 1660s. Madhava's work may have been motivated by his studies in astronomy, since he concentrated mainly on the trigonometric functions. There appears to be no connection between Madhava's school and that of Newton and other European mathematicians. In spite of this, the Keralese and European mathematicians shared some similar methods and results. Both were fascinated with transformation of series, though here they used very different methods.

The mathematician-astronomers of medieval Kerala lived, worked, and taught in large family compounds called illams. Madhava, believed to have been the founder of the school, worked in the Bakulavihara illam in the town of Sangamagrama, a few miles north of Cochin. He was an Emprantiri Brahmin, then considered socially inferior to the dominant Namputiri (or Nambudri) Brahmin. This position does not appear to have curtailed his teaching activities; his most distinguished pupil was Paramesvara, a Namputiri Brahmin. No mathematical works of Madhava have been found, though three of his short treatises on astronomy are extant. The most important of these describes how to accurately determine the position of the moon at any time of the day. Other surviving mathematical works of the Kerala school attribute many very significant results to Madhava. Although his algebraic notation was almost primitive, Madhava's mathematical skill allowed him to carry out highly original and difficult research.

Paramesvara (c.1380–c.1460), Madhava's pupil, was from Asvattagram, about thirty-five miles northeast of Madhava's home town. He belonged to the Vatasreni illam, a famous center for astronomy and mathematics. He made a series of observations of the eclipses of the sun and the moon between 1395 and 1432 and composed several astronomical texts, the last of which was written in the 1450s, near the end of his life. Sankara Variyar attributed to Paramesvara a formula for the radius of a circle in terms of the sides of an inscribed quadrilateral. Paramesvara's son, Damodara, was the teacher of Jyesthadeva (c. 1500–c. 1570) whose works survive and give us all the surviving proofs of this school. Damodara was also the teacher of Nilakantha (c. 1450–c. 1550) who composed the famous treatise called the *Tantrasangraha* (c. 1500), a digest of the mathematical and astronomical knowledge of his time. His works allow us determine his approximate dates since in his *Aryabhatyabhasya*, Nilakantha refers to his observation of solar eclipses in 1467 and 1501. Nilakantha made several efforts to establish new parameters for the mean motions of the planets and vigorously defended the necessity of continually correcting astronomical parameters on the basis of observation. Sankara Variyar (c. 1500–1560) was his student.

The surviving texts containing results on infinite series are Nilakantha's *Tantrasangraha*, a commentary on it by Sankara Variyar called *Yuktidipika*, the *Yuktibhasa* by Jyesthadeva and the *Kriyakramakari*, started by Variyar and completed by his student Mahisamangalam Narayana. All these works are in Sanskrit except the *Yuktibhasa*, written in Malayalam, the language of Kerala. These works provide a summary of major results on series discovered by these original mathematicians of the indistinct past:

A. Series expansions for arctangent, sine, and cosine:

1. $\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots,$
2. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$
3. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots,$
4. $\sin^2\theta = \theta^2 - \frac{\theta^4}{(2^2-2/2)} + \frac{\theta^6}{(2^2-2/2)(3^2-3/2)} - \frac{\theta^8}{(2^2-2/2)(3^2-3/2)(4^2-4/2)} + \cdots.$

In the proofs of these formulas, the range of θ for the first series was $0 \le \theta \le \pi/4$ and for the second and third was $0 \le \theta \le \pi/2$. Although the series for sine and cosine converge for all real values, the concept of periodicity of the trigonometric functions was discovered much later.

B. Series for π:

1. $\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f_i(n+1), \quad i = 1,2,3,$ where

$$f_1(n) = 1/(2n), \quad f_2(n) = n/(2(n^2+1)),$$

and

$$f_3(n) = (n^2+4)/(2n(n^2+5));$$

2. $\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots;$
3. $\frac{\pi}{4} = \frac{4}{1^5+4\cdot 1} - \frac{4}{3^5+4\cdot 3} + \frac{4}{5^5+4\cdot 5} - \cdots;$
4. $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots;$
5. $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{(2\cdot 2^2-1)^2-2^2} + \frac{1}{(2\cdot 4^2-1)^2-4^2} + \frac{1}{(2\cdot 6^2-1)^2-6^2} + \cdots;$
6. $\frac{\pi-2}{4} \approx \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \mp \frac{1}{n^2-1} \pm \frac{1}{2\left((n+1)^2+2\right)}.$

These results were stated in verse form. Thus, the series for sine was described:

> The arc is to be repeatedly multiplied by the square of itself and is to be divided [in order] by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations are to be placed in sequence in a column,

and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the jya (sine) of the arc.

So if r is the radius and s the arc, then the successive terms of the repeated operations mentioned in the description are given by

$$s \cdot \frac{s^2}{(2^2+2)r^2}, \quad s \cdot \frac{s^2}{(2^2+2)r^2} \cdot \frac{s^2}{(4^2+4)r^2}, \dots$$

and the equation is

$$y = s - s \cdot \frac{s^2}{(2^2+2)r^2} + s \cdot \frac{s^2}{(2^2+2)r^2} \cdot \frac{s^2}{(4^2+4)r^2} - \cdots$$

where $y = r\sin(s/r)$. Nilakantha's *Aryabhatiyabhasya* attributes the sine series to Madhava. The *Kriyakramakari* attributes to Madhava the first two cases of B.1, the arctangent series, and series B.4; note that B.4 can be derived from the arctangent by taking $\theta = \pi/6$. The extant manuscripts do not appear to attribute the other series to a particular person. The *Yuktidipika* gives series B.6, including the remainder; it is possible that this series is due to Sankara Variyar, the author of the work. We can safely conclude that the power series for arctangent, sine, and cosine were obtained by Madhava; he is, thus, the first person to express the trigonometric functions as series. In the 1660s, Newton rediscovered the sine and cosine series; in 1671, James Gregory rediscovered the series for arctangent.

The series for $\sin^2\theta$ follows directly from the series for $\cos\theta$ by an application of the double angle formula, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. The series for $\pi/4$ (B.1) has several points of interest. When $n \to \infty$, it is simply the series discovered by Leibniz in 1673. However, this series is not useful for computational purposes because it converges extremely slowly. To make it more effective in this respect, the Madhava school added a rational approximation for the remainder after n terms. They did not explain how they arrived at the three expressions $f_i(n)$ in B.1. However, if we set

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f(n), \tag{1.1}$$

then the remainder $f(n)$ has the continued fraction expansion

$$f(n) = \frac{1}{2} \cdot \frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots, \tag{1.2}$$

when $f(n)$ is assumed to satisfy the functional relation

$$f(n+1) + f(n-1) = \frac{1}{n}. \tag{1.3}$$

The first three convergents of this continued fraction are

$$\frac{1}{2n} = f_1(n), \quad \frac{n}{2(n^2+1)} = f_2(n), \quad \text{and} \quad \frac{1}{2}\frac{n^2+4}{n(n^2+5)} = f_3(n). \tag{1.4}$$

Although this continued fraction is not mentioned in any extant works of the Kerala school, their approximants indicate that they must have known it, at least implicitly. In fact, continued fractions appear in much earlier Indian works. The *Lilavati* of Bhaskara (c. 1150) used continued fractions to solve first-order Diophantine equations and Variyar's *Kriyakramakari* was a commentary on Bhaskara's book.

The approximation in equation B.6 is similar to that in B.1 and gives further evidence that the Kerala mathematicians saw a connection between series and continued fractions. If we write

$$\frac{\pi-2}{4} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \pm \frac{1}{n^2-1} \pm g(n+1), \text{ then} \tag{1.5}$$

$$g(n) = \frac{1}{2n} \cdot \frac{1}{n+} \, \frac{1\cdot 2}{n+} \, \frac{2\cdot 3}{n+} \, \frac{3\cdot 4}{n+} \cdots, \text{ and} \tag{1.6}$$

$$g_1(n) = \frac{1}{2n}, \quad g_2(n) = \frac{1}{2(n^2+2)}. \tag{1.7}$$

Newton, who was very interested in the numerical aspects of series, also found the $f_1(n) = 1/(2n)$ approximation when he saw Leibniz's series. He wrote in a letter of 1676 to Henry Oldenburg:

> By the series of Leibniz also if half the term in the last place be added and some other like device be employed, the computation can be carried to many figures.

Though the accomplishments of Madhava and his followers are quite impressive, the members of the school do not appear to have had any interaction with people outside of the very small region where they lived and worked. By the end of the sixteenth century, the school ceased to produce any further original works. Thus, there appears to be no continuity between the ideas of the Kerala scholars and those outside India or even from other parts of India.

1.2 Transformation of Series

The series in equations B.2 and B.3 are transformations of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

by means of the rational approximations for the remainder. To understand this transformation in modern notation, observe:

$$\frac{\pi}{4} = (1 - f_1(2)) - \left(\frac{1}{3} - f_1(2) - f_1(4)\right) + \left(\frac{1}{5} - f_1(4) - f_1(6)\right) - \cdots. \tag{1.8}$$

The $(n+1)$th term in this series is

$$\frac{1}{2n+1} - f_1(2n) - f_1(2n+2) = \frac{1}{2n+1} - \frac{1}{4n} - \frac{1}{4(n+1)} = \frac{-1}{(2n+1)^3 - (2n+1)}. \tag{1.9}$$

Thus, we arrive at equation B.2. Equation B.3 is similarly obtained:

$$\frac{\pi}{4} = (1 - f_2(2)) - \left(\frac{1}{3} - f_2(2) - f_2(4)\right) + \left(\frac{1}{5} - f_2(4) - f_2(6)\right) - \cdots, \tag{1.10}$$

and here the $(n+1)$th term is

$$\frac{1}{2n+1} - \frac{n}{(2n)^2+1} - \frac{n+1}{(2n+2)^2+1} = \frac{4}{(2n+1)^5 + 4(2n+1)}. \tag{1.11}$$

Clearly, the nth partial sums of these two transformed series can be written as

$$s_i(n) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \mp \frac{1}{2n-1} \pm f_i(2n), \quad i = 1, 2. \tag{1.12}$$

Since series (1.8) and (1.10) are alternating, and the absolute values of the terms are decreasing, it follows that

$$\frac{1}{(2n+1)^3 - (2n+1)} - \frac{1}{(2n+3)^3 - (2n+3)} < \left|\frac{\pi}{4} - s_1(n)\right|$$

$$< \frac{1}{(2n+1)^3 - (2n+1)^3}. \text{ Also,} \tag{1.13}$$

$$\frac{4}{(2n+1)^5 + 4(2n+1)} - \frac{4}{(2n+3)^5 + 4(2n+3)} < \left|\frac{\pi}{4} - s_2(n)\right|$$

$$< \frac{4}{(2n+1)^5 + 4(2n+1)}. \tag{1.14}$$

Thus, taking fifty terms of $1 - \frac{1}{3} + \frac{1}{5} - \cdots$ and using the approximation $f_2(n)$, the last inequality shows that the error in the value of π becomes less than 4×10^{-10}. The Leibniz series with fifty terms is normally accurate in computing π up to only one decimal place; by contrast, the Keralese method of rational approximation of the remainder produces numerically useful results.

1.3 Jyesthadeva on Sums of Powers

The Sanskrit texts of the Kerala school with few exceptions contain merely the statements of results without derivations. It is therefore extremely fortunate that Jyesthadeva's Malayalam text *Yuktibhasa*, containing the methods for obtaining the formulas, has survived. Sankara Variyar's *Yuktidipika* is a modified Sanskrit version of the *Yuktibhasa*. It seems that the *Yuktibhasa* was the text used by Jyesthadeva's students at his illam. From this, one may surmise that Variyar, a student of Nilakantha, also studied with Jyesthadeva whose illam was very close to that of Nilakantha.

A basic result used by the Kerala school in the derivation of their series is that

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p = \frac{1}{p+1}. \tag{1.15}$$

This relation has a long history; sums of powers of integers have been used in the study of area and volume problems at least since Archimedes. Archimedes summed

$$S_n^{(p)} = \sum_{k=1}^{n} k^p$$

for $p = 1$ and $p = 2$. For $p = 2$, he proved the more general result: If $A_1, A_2, \ldots, A_n$ are n lines (we may take them to be numbers) forming an ascending arithmetical progression in which the common difference is equal to A_1 (the least term), then

$$(n+1)A_n^2 + A_1(A_1 + A_2 + \cdots + A_n) = 3(A_1^2 + A_2^2 + \cdots + A_n^2). \tag{1.16}$$

This implies that

$$3(1^2 + 2^2 + \cdots + n^2) = n^2(n+1) + (1 + 2 + 3 + \cdots + n). \tag{1.17}$$

Archimedes used this formula in his work on spirals and in computing the volume of revolution of a segment of a parabola about its axis. The celebrated Arab mathematician al-Haytham (c.965–1039), known also as Alhazen, generalized Archimedes's formula to find the volume of revolution of segment of a parabola about its base. The calculation involved sums of cubes and fourth powers of integers. Al-Haytham proved his generalization by means of a diagram; it can be expressed in modern notation by

$$(n)S_n^{(p)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)}. \tag{1.18}$$

It is interesting that the statement of Jyesthadeva's first lemma leading to the proof of (1.15) is a restatement of al-Haytham's formula; Jyesthadeva's result was stated:

> Whenever we wish to obtain the sum (sankalitam) of any given powers [say the pth powers of natural numbers, up to an assigned limit n], we multiply the sankalitam of the next lower powers [that is, $(p-1)$th powers, up to the given limit n] by the limit [n]. The result will contain the required sankalitam and also the sankalitam of all the sankalitams of all lower powers up to various limits.

Jyesthadeva's next lemma stated:

> Multiply the lower [power] sankalitam [up to the limit of n] by the limit [n]. Subtract from this product the result of dividing the product by an integer one more than the given power [p]. The result will be [asymptotically equal to] the desired sankalitam.

Thus

$$nS_n^{(p-1)}\left(1 - \frac{1}{p+1}\right) \sim S_n^{(p)} \text{ as } n \to \infty. \tag{1.19}$$

Jyesthadeva proved this result inductively, but he did not perform the induction completely. It is easy to see that (1.19) is equivalent to (1.15) and thus Jyesthadeva assumed that

$$S_n^{(p-1)} \sim n^p/p,$$

which is certainly true for $p = 1$. From this it can be deduced that

$$S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_n^{(p-1)} \sim \frac{1^p + 2^p + \cdots + n^p}{p} = \frac{S_n^{(p)}}{p} \text{ as } n \to \infty.$$

Jyesthadeva asserted this but verified it only for $p = 2$ and 3. But once we fill in the gap by proving this for all p, equation (1.18) implies that

$$(n+1)S_n^{(p-1)} \sim S_n^{(p)} + \frac{S_n^{(p)}}{p} \text{ as } n \to \infty.$$

Hence by the inductive hypothesis it follows that

$$S_n^{(p)} \sim \frac{n^{p+1}}{p+1} \text{ as } n \to \infty.$$

This was Jyesthadeva's argument for (1.15).

1.4 Arctangent Series in the *Yuktibhasa*

The following derivation of the arctangent series, attributed to Madhava, boils down to the integration of $1/(1+x^2)$, as do the methods of Gregory and Leibniz.

In Figure 1.1, AC is a quarter circle of radius one with center O; $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta = 1$ and $P_{k-1}P_k = \delta$. EF and $P_{k-1}D$ are perpendicular to OP_k. Now, the triangles OEF and $OP_{k-1}D$ are similar, implying that

$$\frac{EF}{OE} = \frac{P_{k-1}D}{OP_{k-1}} \quad \text{or} \quad EF = \frac{P_{k-1}D}{OP_{k-1}}.$$

The similarity of the triangles $P_{k-1}P_kD$ and OAP_k gives

$$\frac{P_{k-1}P_k}{OP_k} = \frac{P_{k-1}D}{OA} \quad \text{or} \quad P_{k-1}D = \frac{P_{k-1}P_k}{OP_k}.$$

Thus,

$$EF = \frac{P_{k-1}P_k}{OP_{k-1}OP_k} \simeq \frac{P_{k-1}P_k}{OP_k^2} = \frac{P_{k-1}P_k}{1+AP_k^2} = \frac{\delta}{1+k^2\delta^2}.$$

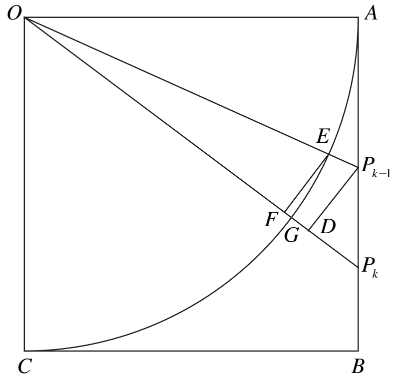

Figure 1.1. Rectifying a circle by the arctangent series.

Now

$$\text{arc } EG \simeq EF \simeq \frac{\delta}{1+k^2\delta^2},$$

and if we write $AP_k = x = \tan\theta$, where $\theta = A\widehat{O}P_k$, then

$$\arctan x = \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\delta}{1+j^2\delta^2}. \tag{1.20}$$

To compute this limit, Jyesthadeva expanded $\frac{1}{1+j^2\delta^2}$ as a geometric series. He derived the series by an iterative procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

Thus, (1.20) is converted to

$$\begin{aligned}\arctan x &= \lim_{k\to\infty}\left(\delta\sum_{j=1}^{k}1 - \delta^3\sum_{j=1}^{k}j^2 + \delta^5\sum_{j=1}^{k}j^4 - \cdots\right)\\ &= \lim_{k\to\infty}\left(\frac{x}{k}\sum_{j=1}^{k}1 - \frac{x^3}{k^3}\sum_{j=1}^{k}j^2 + \frac{x^5}{k^5}\sum_{j=1}^{k}j^4 - \cdots\right)\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.\end{aligned}$$

The last step follows from (1.15). Note that this is the Madhava–Gregory series for $\arctan x$ and the series for $\pi/4$ follows by taking $x = 1$.

1.5 Derivation of the Sine Series in the *Yuktibhasa*

Once again, Madhava's derivation of the sine series has similarities with Leibniz's derivation of the cosine series. In Figure 1.2, suppose that $A\widehat{O}P = \theta$, $OP = R$, P is the midpoint of the arc $P_{-1}P_1$, and PQ is perpendicular to OA, where O is the origin of the coordinate system. Let $P = (x, y)$, $P_1 = (x_1, y_1)$, and $P_{-1} = (x_{-1}, y_{-1})$. From the similarity of the triangles $P_{-1}Q_1P_1$ and OPQ, we have

$$\frac{P_{-1}P_1}{OP} = \frac{x_{-1} - x_1}{y} = \frac{y_1 - y_{-1}}{x}. \tag{1.21}$$

For a small arc $P_{-1}P = \Delta\theta/2$, identified by Jyesthadeva with the line segment $P_{-1}P$, we can write (1.21) as

$$\cos\left(\theta + \frac{\Delta\theta}{2}\right) - \cos\left(\theta - \frac{\Delta\theta}{2}\right) = -\sin\theta\,\Delta\theta \quad \text{and} \tag{1.22}$$

$$\sin\left(\theta + \frac{\Delta\theta}{2}\right) - \sin\left(\theta - \frac{\Delta\theta}{2}\right) = \cos\theta\,\Delta\theta. \tag{1.23}$$

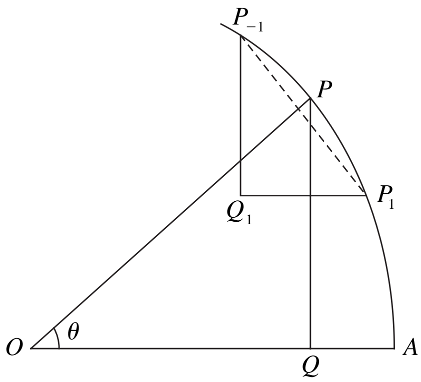

Figure 1.2. Derivation of the sine series.

In fact, Bhaskara earlier stated this last relation and proved it in the same way; he applied it to the discussion of the instantaneous motion of planets. Interestingly, in the 1650s, Pascal used a very similar argument to show that $\int \cos\theta\, d\theta = \sin\theta$ and $\int \sin\theta\, d\theta = -\cos\theta$.

From (1.22) and (1.23) Jyesthadeva derived the result, given in modern notation:

$$\sin\theta - \theta = -\int_0^\theta \int_0^t \sin u\, du dt. \tag{1.24}$$

We also note that Leibniz found the series for cosine using a similar method of repeated integration. In Jyesthadeva, the integrals are replaced by sums and double integrals by sums of sums. The series is then obtained by using successive polynomial approximations for $\sin\theta$. For example, when the first approximation $\sin u \approx u$ is used in the right-hand side of (1.24), the result is

$$\sin\theta - \theta \sim -\frac{\theta^3}{3!} \quad \text{or} \quad \sin\theta \sim \theta - \frac{\theta^3}{3!}.$$

When this approximation is employed in (1.24), we obtain

$$\sin\theta - \theta \sim -\frac{\theta^3}{3!} + \frac{\theta^5}{5!}.$$

Briefly, Jyesthadeva arrived at the sums approximating (1.24) by first dividing AP into n equal parts using division points $P_1, P_2, \ldots, P_{n-1}$. Denote the midpoint of the arc $P_{k-1}P_k$ as $P_{k-1/2}$. Then by (1.21)

$$x_{k+1/2} - x_{k-1/2} = -\frac{\Delta\theta}{2R} y_k, \quad k = 1, 2, \ldots, n-1. \tag{1.25}$$

We also have

$$(y_{k+1} - y_k) - (y_k - y_{k-1}) = \frac{\Delta\theta}{2R}\left(x_{k+1/2} - x_{k-1/2}\right), \quad k = 1, \ldots, n-1 \text{ or} \tag{1.26}$$

$$y_{k+1} - 2y_k + y_{k-1} = -\left(\frac{\Delta\theta}{2R}\right)^2 y_k, \quad k = 1, 2, \ldots, n-1. \tag{1.27}$$

Now start with $k = n-1$ and multiply the equations by $1, 2, \ldots, n-1$ respectively and sum up the resulting equations. We then have

$$\begin{aligned} y_n - ny_1 &= -\left(\frac{\Delta\theta}{2R}\right)^2 (y_{n-1} + 2y_{n-2} + \cdots (n-1)y_1) \\ &= -\left(\frac{\Delta\theta}{2R}\right)^2 (y_1 + (y_1 + y_2) + \cdots + (y_1 + y_2 + \cdots y_{n-1})). \end{aligned} \tag{1.28}$$

This is the result corresponding to (1.24). To obtain the successive polynomial approximations, Jyesthadeva had to work with sums of powers of integers; in order to deal with these sums, he applied the same lemma (1.15) he had used for the arctangent series.

1.6 Continued Fractions

The noted twelfth-century Indian mathematician Bhaskara, who lived and worked in the area now known as Karnataka, used continued fractions in his c. 1150 *Lilavati*. The Kerala school was certainly familiar with Bhaskara's work, since they commented on it. It is therefore possible that they were aware of the specific continued fractions (1.2) and (1.6) for the error terms, even though they mentioned only the first few convergents of these fractions. They did not indicate how they obtained these convergents. Some historians have suggested that Madhava may have found the approximations for the error term, without knowing the continued fractions, by comparing the first few partial sums of the series with a known rational approximation of π. Others speculate that Madhava may have used a method of Wallis.

Whether or not Madhava knew it, Wallis's technique can be used to derive the continued fractions of which the Kerala school gave the convergents; this may be of interest. Start with the functional equation (1.4) for $f(n)$,

$$f(n+1) + f(n-1) = \frac{1}{n}. \tag{1.29}$$

It is obvious that a first approximation for $f(n)$ is given by $f(n) \approx \frac{1}{2n}$. As a first step toward the continued fraction for $f(n)$, set

$$f(n) = \frac{1}{2r_n^{(0)}} \quad \text{and} \quad r_n^{(0)} = n + \frac{1}{r_n^{(1)}}. \tag{1.30}$$

It follows from (1.29) that $r_n^{(0)}$ satisfies

$$\left(2r_{n+1}^{(0)} - n\right)\left(2r_{n-1}^{(0)} - n\right) = n^2. \tag{1.31}$$

From (1.30)

$$2r_{n+1}^{(0)} - n = n + 2 + \frac{2}{r_{n+1}^{(1)}},$$