Classical solutions play an important role in quantum field theory, high energy physics, and cosmology. Real time soliton solutions give rise to particles, such as magnetic monopoles, and extended structures, such as domain walls and cosmic strings, that have implications for the cosmology of the early universe. Imaginary time Euclidean instantons are responsible for important nonperturbative effects, while Euclidean bounce solutions govern transitions between metastable states.

Written for advanced graduate students and researchers in elementary particle physics, cosmology, and related fields, this book brings the reader up to the level of current research in the field. The first half of the book discusses the most important classes of solitons: kinks, vortices, and magnetic monopoles. The cosmological and observational constraints on these are covered, as are more formal aspects, including BPS solitons and their connection with supersymmetry. The second half is devoted to Euclidean solutions, with particular emphasis on Yang–Mills instantons and on bounce solutions.

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To Carolyn, Michael, and Cate
## Contents

*Preface*  
  
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>page xiii</td>
</tr>
<tr>
<td>1.1</td>
<td>Overview</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Conventions</td>
<td>3</td>
</tr>
</tbody>
</table>

| 2 | One-dimensional solitons | 6 |
| 2.1 | Kinks | 6 |
| 2.2 | Quantizing about the kink | 13 |
| 2.3 | Zero modes and collective coordinates | 22 |
| 2.4 | Fermions and fermion zero modes | 24 |
| 2.5 | Kinks in more spacetime dimensions | 27 |
| 2.6 | Multikink dynamics | 29 |
| 2.7 | The sine-Gordon–massive Thirring model equivalence | 34 |

| 3 | Solitons in more dimensions—Vortices and strings | 38 |
| 3.1 | First attempt—global vortices | 38 |
| 3.2 | Derrick’s theorem | 42 |
| 3.3 | Gauged vortices | 44 |
| 3.4 | Multivortex solutions | 47 |
| 3.5 | Quantization and zero modes | 49 |
| 3.6 | Adding fermions | 52 |

| 4 | Some topology | 57 |
| 4.1 | Vacuum manifolds | 57 |
| 4.2 | Homotopy and the fundamental group \( \pi_1(M) \) | 58 |
| 4.3 | Fundamental groups of Lie groups | 61 |
| 4.4 | Vortices and homotopy | 64 |
| 4.5 | Some illustrative vortex examples | 68 |
| 4.6 | Higher homotopy groups | 74 |
| 4.7 | Some results for higher homotopy groups | 77 |

| 5 | Magnetic monopoles with U(1) charges | 81 |
| 5.1 | Magnetic monopoles in electromagnetism | 81 |
| 5.2 | The ’t Hooft–Polyakov monopole | 89 |
x

5.3 Another gauge, another viewpoint 94
5.4 Solutions with higher magnetic charge 96
5.5 Zero modes and dyons 97
5.6 Spin from isospin, fermions from bosons 100
5.7 Fermions and monopoles 104

6 Magnetic monopoles in larger gauge groups 108
6.1 Larger gauge groups—the external view 108
6.2 Larger gauge groups—topology 115
6.2.1 SU(3) broken to SU(2)×U(1) 115
6.2.2 A $Z_2$ monopole 119
6.2.3 A light doubly charged monopole 120
6.2.4 Electroweak monopoles? 121
6.3 Monopoles in grand unified theories 121
6.3.1 SU(5) monopoles 122
6.3.2 SO(10) monopoles 124
6.4 Chromodyons 125
6.5 The Callan–Rubakov effect 128

7 Cosmological implications and experimental bounds 130
7.1 Brief overview of big bang cosmology 130
7.2 Symmetry restoration and cosmological phase transitions 133
7.3 The Kibble mechanism 136
7.4 Gravitational and cosmological consequences of domain walls and strings 139
7.5 Evolution of the primordial monopole abundance 142
7.6 Observational bounds and the primordial monopole problem 145

8 BPS solitons, supersymmetry, and duality 149
8.1 The BPS limit as a limit of couplings 149
8.2 Energy bounds 151
8.3 Supersymmetry 155
8.4 Multisoliton solutions 160
8.5 The moduli space approximation 163
8.6 BPS monopoles in larger gauge groups 166
8.7 Montonen–Olive duality 172

9 Euclidean solutions 175
9.1 Tunneling in one dimension 175
9.2 WKB tunneling with many degrees of freedom 178
9.3 Path integral approach to tunneling: instantons 181
9.4 Path integral approach to tunneling: bounces 186
9.5 Field theory 190
## Contents

### 10 Yang–Mills instantons 192
10.1 $A_0 = 0$ gauge 192
10.2 Yang–Mills vacua: $A_0 = 0$ gauge 194
10.3 Yang–Mills vacuum: axial gauge 201
10.4 Some topology 203
10.5 't Hooft symbols 207
10.6 The unit instanton 209
10.7 Multi-instanton solutions 212
10.8 Counting parameters with an index theorem 213
10.9 Larger gauge groups 220
10.10 The Atiyah–Drinfeld–Hitchin–Manin construction 223
10.11 The ADHM construction for larger gauge groups 228
10.12 One-loop corrections 231

### 11 Instantons, fermions, and physical consequences 236
11.1 Anomalies 236
11.2 Spectral flow and fermion zero modes 239
11.3 QCD and the $U(1)$ problem 245
11.4 Baryon number violation by electroweak processes 246
11.5 CP violation and the $\theta F \tilde{F}$ term 248

### 12 Vacuum decay 254
12.1 Bounces in a scalar field theory 254
12.2 The thin-wall approximation 263
12.3 Evolution of the bubble after nucleation 265
12.4 Tunneling at finite temperature 267
12.5 Including gravity: bounce solutions 272
12.6 Interpretation of the bounce solutions 284
12.7 Curved spacetime evolution after bubble nucleation 291

#### Appendix A: Roots and weights 295
A.1 Root systems 295
A.2 Weights 302

#### Appendix B: Index theorems for BPS solitons 305
B.1 Vortices 306
B.2 Monopoles 308

References 312
Index 324
Preface

Semiclassical methods based on classical solutions play an important role in quantum field theory, high energy physics, and cosmology. Real-time soliton solutions give rise both to new particles, such as magnetic monopoles, and to extended structures, such as domain walls and cosmic strings. These could have been produced as topological defects in the very early universe. Confronting the consequences of such objects with observation and experiment places important constraints on grand unification and other potential theories of high energy physics beyond the standard model. Imaginary-time Euclidean instanton solutions are responsible for important nonperturbative effects. In the context of quantum chromodynamics they resolve one puzzle—the U(1) problem—while raising another—the strong CP problem—whose resolution may entail the existence of a new species of particle, the axion. The Euclidean bounce solutions govern transitions between metastable vacuum states. They determine the rates of bubble nucleation in cosmological first-order transitions and give crucial information about the evolution of these bubbles after nucleation. These bounces become of particular interest if there is a string theory landscape with a myriad of metastable vacua.

This book is intended as a survey and overview of this field. As the title indicates, there is a dual focus. On the one hand, solitons and instantons arise as solutions to classical field equations. The study of their many varieties and their mathematical properties is a fascinating subfield of mathematical physics that is of interest in its own right. Much of the book is devoted to this aspect, explaining how the solutions are discovered, their essential properties, and the topological underpinnings of many of the solutions. However, the physical significance of these classical objects can only be fully understood when they are seen in the context of the corresponding quantum field theories. To that end, there is also a discussion of quantum effects, including those arising from the interplay of fermion fields with topologically nontrivial classical solutions, and of some of the phenomenological consequences of instantons and solitons.

The first half of this book focuses on real-time classical solutions. I focus in particular on three classes of solitons—kinks, vortices, and magnetic monopoles—in one, two, and three spatial dimensions, respectively. Several chapters are devoted to their classical properties and many aspects of their quantum behavior. These are followed by a chapter that discusses the cosmological consequences of domain walls and cosmic strings—the dimensionally extended manifestations of kinks and
vortices—and of magnetic monopoles, and the implications of these for proposed high energy theories. Finally, there is a chapter discussing solitons in the BPS limit, including the connections with supersymmetry and duality.

After considering solitons, I turn to Euclidean solutions. Although these are solutions of classical equations, they are associated with tunneling processes that are truly quantum mechanical phenomena. An introductory chapter presenting an overview of this connection is followed by two chapters on Yang–Mills instantons. The first of these is primarily concerned with the mathematical properties of these solutions and their interpretation in terms of vacuum tunneling. Fermions are introduced in the second chapter, which discusses the physical consequences flowing from the instantons. A final chapter describes the bounce solutions and vacuum transitions.

Of necessity, some topics had to be omitted. In particular, Q-balls, nontopological solitons whose existence is based on the possession of a conserved charge rather than on topology, are not covered, nor are skyrmions, a fascinating class of topological solitons.

My goal has been to make the book accessible to advanced graduate students and other newcomers to the field, but also useful for more experienced researchers. I assume that the reader has had an introductory course in quantum field theory and some familiarity with non-Abelian gauge theories, but only the mathematical background of a typical physics graduate student. The homotopy theory needed to understand the topological underpinnings of the solitons is presented and explained. An appendix discusses roots, weights, and other necessary properties of Lie groups and algebras, building on the familiar results associated with SU(2).

I owe much to the colleagues and students with whom I have collaborated in research in this field. I thank Claude Bernard, Xingang Chen, Norman Christ, Huidong Guo, Alan Guth, Jim Hackworth, Conor Houghton, Roman Jackiw, Tom Kibble, Alex Kusenko, Bum-Hoon Lee, Choonkyu Lee, Hakjoon Lee, Kimyeong Lee, Sang-Hoon Lee, Arthur Lue, Ali Masoumi, Dimitrios Metaxas, Chris Miller, Doug Rajaraman, Alex Ridgway, Jon Rosner, Koenraad Schalm, and Piljin Yi. I am grateful to the late Sidney Coleman, from whom I learned field theory and much more.

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