1
Introduction

1.1 Overview

It is a familiar fact that when a field theory is treated quantum mechanically the wave solutions of the classical theory lead to elementary quanta that have a natural interpretation as particles in the quantum theory. This suggests a one-to-one correspondence between fields and particle species and is the basis for the standard applications of perturbative quantum field theory.

However, many classical field theories have solutions that are already particle-like at the classical level. These are characterized by an energy density that is localized in space and that does not dissipate over time. It is natural to ask whether these "solitons", as they are called, have counterparts in the quantum version of the theory. If so, they would presumably be a new species of particle, quite distinct from the "elementary" particle associated with the wave solutions of the free field theory.

It is instructive to compare the classical size of the soliton with the Compton wavelength that it would have in the quantum theory. If the elementary particles of the theory have masses of order m and a characteristic coupling of order g, one typically finds that the soliton has a classical energy

$$E_{\rm classical} \sim \frac{m}{g} \tag{1.1}$$

and a characteristic spatial size $\ell_{\rm soliton} \sim 1/m$. Hence,

$$\lambda_{\rm Compton} \sim \frac{1}{E_{\rm classical}} \sim g\, \ell_{\rm soliton}. \tag{1.2}$$

(I am using units with $\hbar = 1$.) If the coupling is weak, the Compton wavelength is much less than the classical size, and so we might expect the soliton to survive, perhaps with slight modifications, after quantization.

A possible objection is the stark contrast between the smooth profile of the classical solution and the fuzziness of quantum field theory. It is certainly true

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that the quantum fluctuations of the field are large, even divergent, when the field is measured at very short distances. However, these fluctuations are reduced when the field is averaged over a larger smearing distance. We will see that the same weak-coupling regime that gives $\ell_{\rm soliton} \gg \lambda_{\rm Compton}$ also guarantees the existence of a smearing distance that is both large enough to suppress the quantum fluctuations and small enough that the classical field profile is still evident.

The inverse dependence on the coupling implies that in this weak-coupling regime the soliton mass is large, tending toward infinity as the coupling goes to zero. This explains why the effects of the soliton are not seen in perturbation theory. Nevertheless, once the classical solution is known, perturbative methods can be used to quantize the fields about the soliton and to demonstrate that there is indeed a corresponding one-particle state in the quantum theory. Furthermore, the quantum corrections to the classical energy are calculable and give a mass of the form

$$M_{\rm quantum} = E_{\rm classical}\left(1 + c_1 g + c_2 g^2 + \cdots\right). \tag{1.3}$$

What about the strong-coupling regime? Even though the soliton may still be a solution of the classical field equations, the perturbative analysis of the quantum theory breaks down here, and the arguments for a quantum counterpart to the soliton are no longer so clear-cut. However, a new and striking phenomenon may now come into play. There are examples of theories—a particularly well-known pair being the sine-Gordon and massive Thirring models—that are related by a duality that maps the weak-coupling regime of one onto the strong-coupling regime of the other. The sine-Gordon soliton states correspond to elementary particle states of the massive Thirring model, while the elementary particle of the sine-Gordon model becomes a massive Thirring bound state. One must conclude that there is no intrinsic difference between an elementary particle and a soliton. The distinction between them is simply that one viewpoint or the other is more convenient for calculation in a particular coupling regime.

Although we live in a world with three spatial dimensions (and perhaps some additional hidden ones), it can be instructive to consider solitons in fewer dimensions. The analysis of these toy models is often more tractable and helps elucidate issues of principle. Their solutions can also be trivially extended to higher dimensions, where they acquire new physical significance. A particle-like soliton in one dimension can be interpreted as a planar solution in three dimensions, corresponding to a domain wall. Similarly, a two-dimensional particle-like soliton becomes a line solution, or string, in three dimensions.

One can also consider solitons in more than three spatial dimensions. Of particular interest are those in four dimensions. These could be viewed as particles in a hypothetical world with four spatial dimensions. Alternatively, and more importantly, they can be interpreted as solutions in a Euclideanized version of our four-dimensional spacetime. Such Euclidean solutions, or instantons, have no obvious physical significance in a classical context. However, they become

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meaningful quantum mechanically because wavefunctions extend into classically forbidden regions where the potential energy is greater than the total energy. Roughly speaking, one can view this as implying a negative kinetic energy, corresponding to evolution in a Euclidean spacetime with imaginary time. A well-known consequence is that quantum systems can tunnel though potential energy barriers to effect transitions that would be classically forbidden. This leads to important and unexpected nonperturbative effects in gauge theories, with magnitudes that are determined by the action of the relevant instanton. A further result of tunneling processes in field theory is the decay of metastable vacua by bubble nucleation, a process of considerable importance for cosmology. The Euclidean solutions that govern such bubble nucleation are known as bounces.

Finally, a note on terminology. I follow the practice in high energy physics of using the term soliton for any localized classical solution that does not dissipate over time. However, the reader should be aware that some other fields use a more restrictive definition, with the term only used for solutions, arising in integrable systems, that emerge from scattering processes without deformation or loss of energy.

1.2 Conventions

Metric and indices

For the spacetime metric I use the "mostly minus" convention, with the metric $\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$ in flat four-dimensional spacetime. Coordinates are defined by

$$x^\mu = (t, x, y, z) = (t, \mathbf{x}) \tag{1.4}$$

so that

$$\partial_\mu = (\partial/\partial t, \boldsymbol{\nabla}). \tag{1.5}$$

Lorentzian spacetime indices are denoted by Greek letters and summation over repeated indices, one upper and one lower, is to be understood. Purely spatial indices are denoted by Latin letters, generally from the middle of the alphabet; summation over repeated indices (possibly both upper or both lower) is also to be understood. Euclidean spacetime indices are denoted by Latin letters.

The antisymmetric tensor in any dimension is defined to be unity when all of its indices are upper and in numerical order. Thus, $\epsilon^{123} = \epsilon^{0123} = \epsilon^{1234} = 1$.

Dirac matrices

The Dirac matrices in four-dimensional Lorentzian spacetime obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1.6}$$

Of these, γ^0 is Hermitian, while the remaining three are anti-Hermitian. The matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.7}$$

is Hermitian and obeys $(\gamma^5)^2 = I$.

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Units

I use natural units with c, $\hbar$, and Boltzmann's constant k_B all equal to unity.

Gauge fields

Conventions associated with gauge fields vary within the soliton and instanton literature. Those used in this book are described below.

The electromagnetic potential is

$$A^\mu = (\Phi, \mathbf{A}) \tag{1.8}$$

where Φ and $\mathbf{A}$ are the usual scalar and vector potentials. The field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.9}$$

so that, e.g., $F_{12} = F^{12} = -B_z$ and $F_{03} = F^{30} = E_z$. The covariant derivative of a complex field carrying electromagnetic [or any other U(1)] charge q is given by

$$D_\mu \phi = (\partial_\mu + iqA_\mu)\phi. \tag{1.10}$$

The Lagrangian is then invariant under U(1) gauge transformations of the form

$$\begin{aligned} \phi &\to e^{iq\Lambda(x)}\phi, \\ A_\mu &\to A_\mu - \partial_\mu \Lambda(x). \end{aligned} \tag{1.11}$$

In non-Abelian gauge theories the gauge field is written as a Hermitian element of the Lie algebra

$$A_\mu = A_\mu^a T^a, \tag{1.12}$$

where the Hermitian generators T^a are normalized so that

$$\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}. \tag{1.13}$$

They obey

$$[T^a, T^b] = i f_{abc} T^c, \tag{1.14}$$

with the structure constants f_{abc} being totally antisymmetric. This corresponds to the standard normalization for the fundamental representation of SU(2), with the generators being $\sigma^a/2$, where the σ^a are the Pauli matrices. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{1.15}$$

with components

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c. \tag{1.16}$$

A matter field ϕ can be written as a column vector transforming under an irreducible representation of the gauge group. Its covariant derivative is

$$D_\mu \phi = \partial_\mu \phi - igA_\mu \phi. \tag{1.17}$$

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With components written out explicitly, this is

$$(D_\mu \phi)_j = \partial_\mu \phi_j - igA^a_\mu (t^a)_{jk}\phi_k \tag{1.18}$$

with i, j, and k running from 1 to N and $(t^a)_{jk}$ denoting the appropriate representation of the generators.

Under a non-Abelian gauge transformation $U(x)$, the various quantities above transform as

$$\begin{aligned} A_\mu &\longrightarrow UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} = UA_\mu U^{-1} + \frac{i}{g}U\,\partial_\mu U^{-1}, \\ F_{\mu\nu} &\longrightarrow UF_{\mu\nu}U^{-1}, \\ \phi &\longrightarrow \mathcal{U}\phi. \end{aligned} \tag{1.19}$$

where $\mathcal{U}$ is the transformation written in the appropriate representation of the group. For an infinitesimal gauge transformation

$$U = e^{i\Lambda} \approx I + i\Lambda + \cdots \tag{1.20}$$

the change in the gauge potential is

$$\delta A_\mu = \frac{1}{g}\partial_\mu \Lambda - i[A_\mu, \Lambda] = \frac{1}{g}D_\mu \Lambda. \tag{1.21}$$

If the matter fields transform under the adjoint representation, an alternative notation is to write them as linear combinations of the generators,

$$\phi = \phi^a T^a \tag{1.22}$$

with

$$D_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi]. \tag{1.23}$$

In the special case of a triplet field in an SU(2) gauge theory (where $f_{abc} = \epsilon_{abc}$) I sometimes adopt the standard three-dimensional vector notation and write

$$\begin{aligned} D_\mu \boldsymbol{\phi} &= \partial_\mu \boldsymbol{\phi} + g\mathbf{A}_\mu \times \boldsymbol{\phi}, \\ \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu. \end{aligned} \tag{1.24}$$

It is sometimes convenient to absorb the gauge coupling in the gauge field by a rescaling $A_\mu \to gA_\mu$. The Yang–Mills Lagrangian is then

$$\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu a} = -\frac{1}{2g^2}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}. \tag{1.25}$$

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2
One-dimensional solitons

Field theories in one spatial dimension provide a natural starting point for the study of solitons. Because of the simplifications that result from working in one dimension, many more calculations can be carried through explicitly. In addition, the topological considerations that play an important role in all dimensions are particularly easy to visualize in one-dimensional theories. Although the primary value of these theories is as toy models, some of the results we will obtain find application in the real world. First, there are condensed matter systems that can be treated as essentially one-dimensional, some of which support solitons. Second, some of the one-dimensional solitons that we will find can be trivially extended to higher dimensions, so that a localized one-dimensional soliton can become a planar domain wall in higher dimensions.

2.1 Kinks

The classic example of a soliton in one spatial dimension [1, 2] arises in a theory with a single scalar field ϕ and Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \tag{2.1}$$

where the scalar field potential

$$\begin{aligned} V(\phi) &= -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}v^4 \\ &= \frac{\lambda}{4}(\phi^2 - v^2)^2. \end{aligned} \tag{2.2}$$

Here m^2 and λ are both positive and

$$v = \sqrt{\frac{m^2}{\lambda}}. \tag{2.3}$$

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For later reference, note that in two spacetime dimensions the scalar field is dimensionless, so that the coupling constants are dimensionful. The dimensionless parameter that signals weak or strong coupling is the ratio λ/m^2.

The Lagrangian is invariant under the transformation $\phi \to -\phi$. However, this symmetry is spontaneously broken, with $V(\phi)$ having two degenerate minima, at $\phi = \pm v$. The constant term in Eq. (2.2), which has no effect on the dynamics, was chosen so that $V = 0$ at these minima. When the theory is quantized, these two minima correspond to two physically equivalent vacua. Choosing either one of them, say $\phi = v$, and then expanding in terms of the shifted field $\phi - v$, one finds that the theory has a single elementary scalar particle, with mass $\sqrt{2}\,m$.

The classical Euler–Lagrange equation of the theory is

$$\frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} = -\lambda(\phi^2 - v^2)\phi. \tag{2.4}$$

We are particularly interested in static solutions, which satisfy

$$0 = \frac{d^2\phi}{dx^2} - \lambda(\phi^2 - v^2)\phi. \tag{2.5}$$

This is a nonlinear equation, and it may not be obvious from the outset that it has any nonsingular solutions other than the two constant vacuum solutions, $\phi(x) = v$ and $\phi(x) = -v$. To persuade ourselves that it does, note that Eq. (2.5) is also the condition for a configuration $\phi(x)$ to be a stationary point of the potential energy[1]

$$U[\phi(x)] = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right]. \tag{2.6}$$

This is not surprising. For a system whose kinetic energy is purely quadratic in time derivatives, the static solutions of the equations of motion are just the stationary points of the potential energy; the stable solutions are given by the local minima of U.

Thus, our task is to show that there are configurations other than the vacuum solutions that are local minima of $U[\phi(x)]$. To this end, consider a configuration, such as the one shown in Fig. 2.1, in which $\phi(\infty) = v$ and $\phi(-\infty) = -v$. Unless we have made a remarkably lucky choice, this configuration will not be a solution. This means that it can be smoothly varied in such a way as to lower its potential energy. Continuing this process until a minimum of U is reached will lead us to a static solution. Because a smooth variation cannot change the values of ϕ at spatial infinity, the solution we are led to will have $\phi(\infty) \neq \phi(-\infty)$, and so

[1] It is important to remember that the potential energy includes not just the contribution from the scalar field potential $V(\phi)$, but also that from the spatial gradient terms.

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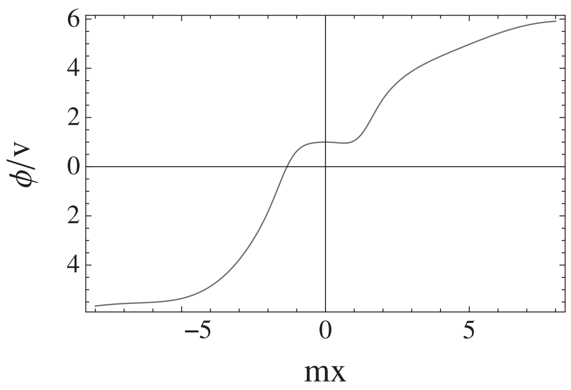

Fig. 2.1. A field configuration that cannot be smoothly deformed to a vacuum solution.

cannot be either of the vacuum solutions. It must instead be a nontrivial spatially varying solution; i.e., the soliton that we are seeking.[2]

Some important properties of this solution are revealed by a rescaling of variables. If we write $\phi = vf$ and $u = mx$, Eq. (2.5) becomes

$$0 = \frac{d^2 f}{du^2} - f(f^2 - 1), \tag{2.7}$$

with $f(\pm\infty) = \pm 1$, while the energy of this static solution takes the form

$$E = \frac{m^3}{\lambda} \int du \left[\frac{1}{2} \left(\frac{df}{du} \right)^2 + \frac{1}{4}(f^2 - 1)^2 \right]. \tag{2.8}$$

It is evident from Eq. (2.7) that $f(u)$ does not contain any explicit factors of m or λ. Hence, its spatial variation is characterized by a distance that is of order unity when measured in terms of u, and thus of order m^{-1} when measured in terms of x. Because the integral on the right-hand side of Eq. (2.8) is also independent of m and λ, it must be of order unity, so the solution has an energy of order m^3/λ.

This is much greater than the mass $\sqrt{2}\,m$ of the elementary scalar when the coupling is weak, and diverges in the limit $\lambda \to 0$. This fact, which is characteristic of solitons in field theory, explains why solitons are not encountered in ordinary perturbative approaches to quantum field theory.

[2] This argument is not rigorous, and must be used with care. Because the space of field configurations is not compact, there need not be any configuration that minimizes the potential energy. Although this does not happen here, we will encounter such a situation when we discuss multisoliton configurations later in this section.

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It is now time to tackle the field equation directly. Multiplying both sides of Eq. (2.5) by $d\phi/dx$ gives

$$0 = \frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - \frac{\lambda}{4}(\phi^2 - v^2)^2\right]. \tag{2.9}$$

Hence, the quantity in brackets must be independent of x. Evaluating it at $x = \infty$, we see that it actually vanishes. It follows that

$$\frac{d\phi}{dx} = \pm\sqrt{\frac{\lambda}{2}}(\phi^2 - v^2). \tag{2.10}$$

Our boundary conditions require the upper sign. Straightforward integration then gives

$$\phi(x) = v \tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.11}$$

where x_0 is a constant of integration. This solution, which is shown in Fig. 2.2, is known as the kink solution; x_0 can be viewed as specifying the position of the kink. The solution obtained by starting with the opposite boundary conditions and taking the lower sign in Eq. (2.10),

$$\phi(x) = -v \tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.12}$$

is called the antikink.

The energy density, shown in Fig. 2.3, is the same for both solutions. It is concentrated within a region of width $\sim m^{-1}$ centered about x_0. Outside this region, the field is essentially indistinguishable from that in a vacuum. Although it is a different vacuum on opposite sides of the kink, this is not evident to a local

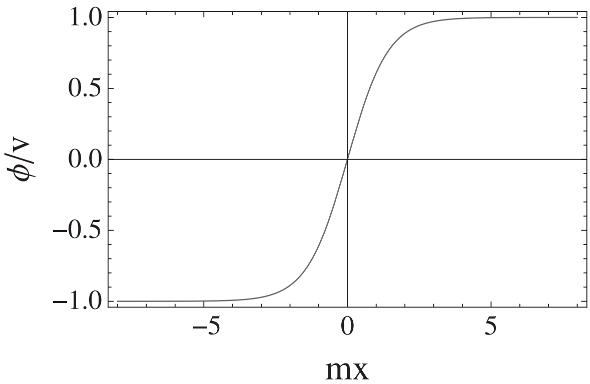

Fig. 2.2. The kink solution of Eq. (2.11), with $x_0 = 0$.

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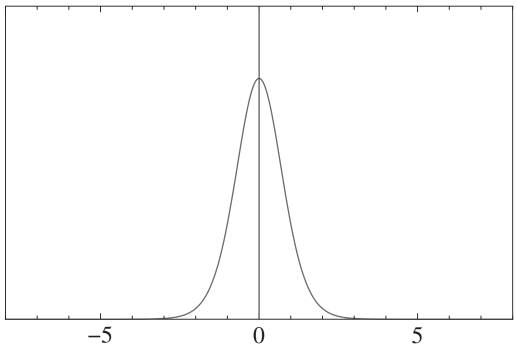

Fig. 2.3. The energy density of the kink solution with $x_0 = 0$.

observer. This localization of the energy suggests that the kink be interpreted as a kind of particle, with its mass given by the total energy of the static solution,

$$M_{\rm cl} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{2.13}$$

The subscript here is intended to indicate that this is just the classical approximation to the mass. We will see in the next section that there are quantum corrections to the mass.

If the kink is to be interpreted as a particle, then there should also be solutions corresponding to moving kinks. Lorentz transforming the static solution gives

$$\phi(x,t) = v \tanh\left[\frac{m}{\sqrt{2}}\frac{(x-ut-x_0)}{\sqrt{1-u^2}}\right], \tag{2.14}$$

which describes a Lorentz-contracted kink moving with velocity u. The energy of this solution,

$$\begin{aligned} E &= \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] \\ &= \frac{M_{\rm cl}}{\sqrt{1-u^2}}, \end{aligned} \tag{2.15}$$

is precisely what is required for a particle with mass $M_{\rm cl}$ moving with velocity u.

The key element for establishing the existence of the kink was that $V(\phi)$ had multiple degenerate vacua, and that the field approached different vacua at the two points of spatial infinity. Because of this intertwining of the topology of the vacua with the topology of spatial infinity, the resulting solitons are known as topological solitons. We can define a topological current

$$J^\mu_{\rm top} = \frac{1}{2v}\,\epsilon^{\mu\nu}\partial_\nu\phi, \tag{2.16}$$

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